

Optimal stopping of a Hilbert space valued diffusion: an infinite dimensional variational inequality*

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July 4, 2012

Abstract

A finite horizon optimal stopping problem for an infinite dimensional diffusion X is analyzed by means of variational techniques. The diffusion is driven by a SDE on a Hilbert space \mathcal{H} with a non-linear diffusion coefficient $\sigma(X)$ and a generic unbounded operator A in the drift term. When the gain function Θ is time-dependent and fulfils mild regularity assumptions, the value function \mathcal{U} of the optimal stopping problem is shown to solve an infinite-dimensional, parabolic, degenerate variational inequality on an unbounded domain. Once the coefficient $\sigma(X)$ is specified, the solution of the variational problem is found in a suitable Banach space \mathcal{V} fully characterized in terms of a Gaussian measure μ .

This work provides the infinite-dimensional counterpart, in the spirit of Bensoussan and Lions [3], of well-known results on optimal stopping theory and variational inequalities in \mathbb{R}^n . These results may be useful in several fields, as in mathematical finance when pricing American options in the HJM model.

MSC2010 Classification: 60G40, 49J40, 35R15.

Key words: optimal stopping, infinite-dimensional stochastic analysis, parabolic partial differential equations, degenerate variational inequalities.

1 Introduction

This paper studies a finite horizon optimal stopping problem associated to an infinite-dimensional diffusion process by means of variational techniques. It is well known that the value function of a wide class of optimal stopping problems for general diffusions in \mathbb{R}^n may be characterized as the solution of a suitable variational problem (see [3] and references therein for a survey). In the language of PDE this amounts to solve a free-boundary problem and, when possible to provide a description of the free-boundary itself. The free-boundary makes the state space split into two regions: the region where it is optimal to let the diffusion evolve (continuation region) and its complement, where it is optimal to stop at once (stopping region) (cf. [29] and references therein for an extensive exposition). Hence an explicit stopping strategy for the optimal stopping problem is found in terms of the free-boundary.

This work is motivated by a central problem in the modern theory of mathematical finance. In fact, pricing American bond options on the forward interest rate curve gives rise to an infinite

*These results extend a portion of the second Author PhD dissertation [12] under the supervision of the first Author.

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dimensional optimal stopping problem. This is a consequence of the dependence of the bond's price on the whole structure of the forward curve. The results obtained here will be extended to solve that particular financial problem in a forthcoming paper [7].

Optimal stopping for processes in locally compact spaces has attracted enormous attention in the last decades (cf. [14], [31], [35] among others) while the case of general infinite-dimensional Markov processes has been studied in relatively few papers. The earliest paper on infinite dimensional optimal stopping and variational inequalities we are aware of is [8]. There Chow and Menaldi extended known finite dimensional results, in the spirit of [3], to the case of a particular infinite dimensional linear diffusion.

A first attempt towards a more comprehensive study of optimal stopping theory for processes taking values in a Polish space was made by J. Zabczyk [36] in 1997 from a purely probabilistic point of view and later on, in 2001, by variational methods [37]. Recently Barbu and Marinelli [2] contributed further insights in this direction adopting arguments similar to those in Zabczyk's works. In both [2] and [37] the Authors considered a diffusion process on a functional space \mathcal{H} and solved the variational problem in a suitable L^2 -space with respect to a measure on \mathcal{H} . The solution was characterized in a mild sense, adopting the general theory of monotone operators and associated semigroups (cf. [5]).

A different approach is based on viscosity theory. It is extensively exploited to solve general stochastic control problems (cf. [16] for a survey) and the infinite-dimensional case is currently the object of intense study (cf. [20], [21], [22], [33] among others). However, as far as we know, the only paper on infinite-dimensional variational inequalities related to optimal stopping problems studied by viscosity methods is [18] by D. Gałtarek and A. Święch. The Authors deal with a problem arising in finance. They characterize the value function of the optimal stopping problem when the underlying diffusion has a particular form not involving the unbounded term normally arising in infinite-dimensional stochastic differential equations (cf. [10] for a survey).

It is worth mentioning that attempts to provide some numerical results for this class of problems were recently made in [19] and [24]. However, the analysis in [19] relies on the assumption of C^{12} -regularity (in time-space) for the value function and that is hard to verify. On the other hand, in [24] arguments are mostly heuristic, proofs are only sketched and some of them seem incorrect.

In the present paper the underlying process X lives in a general Hilbert space \mathcal{H} and is governed by the SDE (2.2) below with a generic unbounded operator A (which is not even required to be self-adjoint). Under mild regularity assumptions on the gain function Θ , the value function \mathcal{U} of the corresponding optimal stopping problem (see (2.7) below) solves an infinite dimensional variational inequality that is highly degenerate, parabolic and on an unbounded domain. We point out that degenerate variational inequalities represent non-standard problems in the context of PDE theory even at the finite dimensional level (cf. [32]). For the associated optimal stopping problems one may consult the work of J.L. Menaldi [25], [26]. In our case we show that \mathcal{U} solves a variational inequality in a suitable Banach space \mathcal{V} relying on the notion of infinite-dimensional Sobolev space with respect to a Gaussian measure μ . The choice of the measure μ is completely characterized once the non-linear diffusion coefficient σ is specified in (2.2) and fulfills mild regularity assumptions. It is also shown that the first time that \mathcal{U} equals Θ is an optimal stopping time.

This work is ideally the extension of [8] to general diffusions in Hilbert spaces and it provides the infinite dimensional analogue of the results in [25], [26]. It is worth mentioning that under such a wide generality for the coefficients of (2.2) we could not prove uniqueness of the solution to the variational inequality. The uniqueness result in [8] is in fact crucially linked to the very particular choice of the diffusion whilst the one in [2] and [37] relies on the adoption of a measure μ which is *excessive* for the semigroup associated to X . The latter approach requires a formulation of the variational problem in a weaker form than the one proposed here. Also, the

measure μ is only known explicitly in a relatively small number of cases, therefore in the general the meaning of the variational problem remains quite obscure.

The paper is organized as follows. In Section 2 we set the problem and we make the main regularity assumptions on the diffusion X and on the gain function Θ . Then we obtain regularity of the value function \mathcal{U} . Section 3 deals with the approximation of the SDE (2.2) and of the optimal stopping problem (2.7). The SDE is approximated in two steps: first the unbounded term A is replaced by its Yosida approximation A_α , $\alpha > 0$, and afterwards a n -dimensional reduction of the SDE is obtained. In this approximation procedure the corresponding process $X^{(\alpha);n}$ gives rise to an optimal stopping problem whose value function we denote by $\mathcal{U}_\alpha^{(n)}$. By means of purely probabilistic arguments we show that $\mathcal{U}_\alpha^{(n)}$ converges to the value function \mathcal{U} of the original optimal stopping problem for $n \rightarrow \infty$ and $\alpha \rightarrow \infty$. In Section 4 we prove that the value function $\mathcal{U}_\alpha^{(n)}$ is solution of a suitable variational inequality in \mathbb{R}^n and we characterize an optimal stopping time. Section 5 represents the core of this work and it is entirely devoted to prove that our original value function \mathcal{U} solves a suitable infinite-dimensional variational problem. The result is obtained by taking the limit as $n \rightarrow \infty$ and $\alpha \rightarrow \infty$ of the variational problem detailed in Section 4. Both analytical and probabilistic tools are adopted to carry out the proofs and to characterize an optimal stopping time.

The paper is completed by technical Appendices. Appendix A contains the proof of a Lemma of Section 3. Appendix B outlines some results for variational inequalities in bounded domains of \mathbb{R}^n . Appendix C shows the link between a particular class of obstacle problems and optimal stopping theory. Finally, Appendix D contains some compactness results for functions in Gauss Sobolev spaces.

2 Setting and preliminary estimates

Let \mathcal{H} represent a separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and induced norm $\| \cdot \|_{\mathcal{H}}$. Let $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ be the infinitesimal generator of a strongly continuous semigroup of operators $\{S(t), t \geq 0\}$ on \mathcal{H} , cf. [28], where $D(A)$ denotes its domain. Notice that $D(A)$ is dense in \mathcal{H} . Let $\{\varphi_1, \varphi_2, \dots\}$ be an orthonormal basis of \mathcal{H} with $\varphi_i \in D(A)$, $i = 1, 2, \dots$. We now introduce a trace class operator which will be crucial in the following analysis.

Definition 2.1. Let $Q : \mathcal{H} \rightarrow \mathcal{H}$ be the positive, linear operator defined by

$$Q\varphi_i = \lambda_i \varphi_i, \quad \lambda_i > 0, \quad i = 1, 2, \dots, \quad (2.1)$$

and such that $\sum_{i=1}^{\infty} \lambda_i < \infty$, i.e. it is of trace class.

We consider a stochastic framework. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $W := (W^0, W^1, W^2, \dots)$ be an infinite dimensional standard Brownian motion on it. The filtration generated by the Brownian motion is $\{\mathcal{F}_t, t \geq 0\}$ and it is completed by the null sets. Given a fixed finite horizon $S > 0$, in \mathcal{H} we consider the stochastic differential equation (SDE)

$$\begin{cases} dX_t = AX_t dt + \sigma(X_t) dW_t^0, & t \in [0, S], \\ X_0 = x, \end{cases} \quad (2.2)$$

and we denote its unique mild solution by X (cf. [10]). The infinite dimensional Brownian motion W will be needed in what follows to find finite dimensional approximations of X , each driven by a SDE similar to (2.2) but with Brownian motion $\overline{W}^{(n)} := (W^0, \dots, W^n)^\top$ rather than W^0 . We make the following

Assumption 2.2. The map $\sigma : \mathcal{H} \rightarrow \mathcal{H}$ is continuous and

- (1) $\sigma(x) \in Q(\mathcal{H})$, $\forall x \in \mathcal{H}$, i.e. there exists $\gamma : \mathcal{H} \rightarrow \mathcal{H}$ such that $\sigma(x) = Q\gamma(x)$,
- (2) $\gamma \in C_b^2(\mathcal{H}; \mathcal{H})$.

Assumption 2.2 guarantees existence and uniqueness of a mild solution to (2.2). The assumption (2) might be substantially relaxed, however (1) is crucial for our purposes (cf. Section 5). From now on we denote by X^x the solution of (2.2) starting from x at time zero and by $X^{t,x}$ the one starting from x at time t .

Assumption 2.3. *The covariance operator Q of (2.1) is such that*

$$\sum_{j=1}^{\infty} \|A\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j} < \infty. \quad (2.3)$$

Assumption 2.3 implies that $Q^{\frac{1}{2}}(\mathcal{H}) \subset D(A)$ and hence σ maps \mathcal{H} into a subspace of $D(A^2)$ by Assumption 2.2.

Let $\Theta : [0, S] \times \mathcal{H} \rightarrow \mathbb{R}$ be a non-negative, uniformly bounded function with

$$\sup_{(t,x) \in [0,S] \times \mathcal{H}} \Theta(t, x) \leq \bar{\Theta} \in \mathbb{R}. \quad (2.4)$$

Denote by $C_b^{1,2}([0, T] \times \mathcal{H})$ the set of bounded continuous functions which are continuously differentiable once with respect to time and twice with respect to the space variable (in the Frechét sense) with bounded derivatives. Let $D\Theta : \mathcal{H} \rightarrow \mathcal{H}^*$ denote the Frechét derivative of Θ with respect to the space variable and assume the following regularity for Θ .

Assumption 2.4. $\Theta \in C_b^{1,2}([0, T] \times \mathcal{H})$ and in particular

$$\|D\Theta(t, x)\|_{\mathcal{H}^*} \leq L_{\Theta} \quad t \in [0, S], x \in \mathcal{H}, \quad (2.5)$$

$$\left| \frac{\partial \Theta}{\partial t}(t, x) \right| \leq L'_{\Theta} \quad x \in \mathcal{H}, 0 \leq t \leq S, \quad (2.6)$$

where L_{Θ}, L'_{Θ} are given positive constants.

Remark 2.5. *The regularity of Θ may be substantially weakened without altering the results of this work (cf. Chiarolla and De Angelis [7]).*

In what follows condition (2.5) will be often recalled as Lipschitz property of the gain function. Given $T \leq S$ we aim to study the infinite dimensional optimal stopping problem

$$\mathcal{U}(t, x) := \sup_{t \leq \tau \leq T} \mathbb{E} \{ \Theta(\tau, X_{\tau}^{t,x}) \}, \quad (2.7)$$

with τ a stopping time with respect to the filtration $\{\mathcal{F}_t, t \in [0, S]\}$. This problem has been studied by several Authors (see, for example, [2], [8], [18], [36], [37]). Here we propose a completely new algorithm for the characterization of the value function \mathcal{U} . Our results hold even in the case of a discounted gain function, when the discount factor is a Lipschitz-continuous, non-negative function of X .

We obtain here some preliminary estimates and some regularity properties of the value function \mathcal{U} .

Lemma 2.6. *Let X^x and X^y be the mild solutions of (2.2) starting at x and y , respectively. Then*

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq S} \|X_t^x\|_{\mathcal{H}}^p \right\} \leq C_{p,S}(1 + \|x\|_{\mathcal{H}}^p) \quad 1 \leq p < \infty, \quad (2.8)$$

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq S} \|X_t^x - X_t^y\|_{\mathcal{H}} \right\} \leq C_{p,S} \|x - y\|_{\mathcal{H}}, \quad (2.9)$$

where the positive constant $C_{p,S}$ depends only on p and S .

Proof. The proof of (2.8) follows from [10], Theorem 7.4, whereas the proof of (2.9) is a consequence of [10], Theorem 9.1 and a simple application of Jensen's inequality. \square

Proposition 2.7. *The value function $\mathcal{U}(t, x)$ is non-negative, uniformly bounded with the same upper bound of Θ , i.e.*

$$\sup_{(t,x) \in [0,T] \times \mathcal{H}} \mathcal{U}(t, x) \leq \bar{\Theta}. \quad (2.10)$$

Moreover, there exists $L_{\mathcal{U}} > 0$ such that

$$|\mathcal{U}(t, x) - \mathcal{U}(t, y)| \leq L_{\mathcal{U}} \|x - y\|_{\mathcal{H}}, \quad t \in [0, T], x, y \in \mathcal{H}. \quad (2.11)$$

Proof. The first claim is obvious. To show (2.11) take $x, y \in \mathcal{H}$ and fix $t \in [0, T]$. Then

$$\begin{aligned} \mathcal{U}(t, x) - \mathcal{U}(t, y) &\leq \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ \Theta(\tau, X_{\tau}^{t,x}) - \Theta(\tau, X_{\tau}^{t,y}) \right\} \\ &\leq \mathbb{E} \left\{ \sup_{t \leq s \leq T} |\Theta(s, X_s^{t,x}) - \Theta(s, X_s^{t,y})| \right\} \leq L_{\Theta} \mathbb{E} \left\{ \sup_{t \leq s \leq T} \|X_s^{t,x} - X_s^{t,y}\|_{\mathcal{H}} \right\}, \end{aligned}$$

by (2.5). Similarly for $\mathcal{U}(t, y) - \mathcal{U}(t, x)$; hence

$$|\mathcal{U}(t, x) - \mathcal{U}(t, y)| \leq L_{\Theta} \mathbb{E} \left\{ \sup_{t \leq s \leq T} \|X_s^{t,x} - X_s^{t,y}\|_{\mathcal{H}} \right\}.$$

The coefficients in (2.2) are time-homogeneous, hence

$$\begin{aligned} \mathbb{E} \left\{ \sup_{t \leq s \leq T} \|X_s^{t,x} - X_s^{t,y}\|_{\mathcal{H}} \right\} &= \mathbb{E} \left\{ \sup_{0 \leq s \leq T-t} \|X_s^x - X_s^y\|_{\mathcal{H}} \right\} \\ &\leq \mathbb{E} \left\{ \sup_{0 \leq s \leq S} \|X_s^x - X_s^y\|_{\mathcal{H}} \right\} \leq C_S \|x - y\|_{\mathcal{H}}, \end{aligned}$$

and (2.11) follows with $L_{\mathcal{U}} = L_{\Theta} C_S$. \square

Recall the operator Q of Definition 2.1. Define the centered Gaussian measure μ with covariance operator Q (cf. [4], [9], [11]); that is the restriction to the vectors $x \in \ell_2$ of the infinite product measure

$$\mu(dx) = \prod_{i=1}^{\infty} \frac{1}{\sqrt{2\pi\lambda_i}} e^{-\frac{x_i^2}{2\lambda_i}} dx_i.$$

Let $1 \leq p < \infty$ and consider $f : \mathcal{H} \rightarrow \mathbb{R}$. Define the $L^p(\mathcal{H}, \mu)$ norm as

$$\|f\|_{L^p(\mathcal{H}, \mu)}^p := \int_{\mathcal{H}} |f(x)|^p \mu(dx). \quad (2.12)$$

If $Df : \mathcal{H} \rightarrow \mathcal{H}^*$ is the Frechét derivative of f and \mathcal{H} is identified with its dual, then the $L^2(\mathcal{H}, \mu; \mathcal{H})$ norm of Df is defined as

$$\|Df\|_{L^2(\mathcal{H}, \mu; \mathcal{H})}^2 := \int_{\mathcal{H}} \|Df(x)\|_{\mathcal{H}}^2 \mu(dx). \quad (2.13)$$

Let \overline{D} denote the closure of D in $L^2(\mathcal{H}, \mu)$ (cf. [9]) and let $W^{1,2}(\mathcal{H}, \mu)$ be the Sobolev space defined by

$$W^{1,2}(\mathcal{H}, \mu) := \{f : f \in L^2(\mathcal{H}, \mu), \overline{D}f \in L^2(\mathcal{H}, \mu; \mathcal{H})\}.$$

Notice however that in what follows derivatives are mostly generalized derivatives, hence there is no ambiguity in using D rather than \overline{D} . For $n \in \mathbb{N}$ the finite dimensional counterpart of μ , $L^p(\mathcal{H}, \mu)$ and $L^2(\mathcal{H}, \mu; \mathcal{H})$ are, respectively,

$$\mu_n(dx) := \prod_{i=1}^n \frac{1}{\sqrt{2\pi\lambda_i}} e^{-\frac{x_i^2}{2\lambda_i}} dx_i, \quad L^p(\mathbb{R}^n, \mu_n) \text{ and } L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n).$$

Remark 2.8. Notice that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$, then

$$\|f\|_{L^p(\mathcal{H}, \mu)}^p = \int_{\mathcal{H}} |f(x)|^p \mu(dx) = \int_{\mathbb{R}^n} |f(x)|^p \mu_n(dx) =: \|f\|_{L^p(\mathbb{R}^n, \mu_n)}^p$$

and

$$\|Df\|_{L^2(\mathcal{H}, \mu; \mathcal{H})}^2 = \int_{\mathcal{H}} \|Df(x)\|_{\mathcal{H}}^2 \mu(dx) = \int_{\mathbb{R}^n} \|Df(x)\|_{\mathbb{R}^n}^2 \mu_n(dx) =: \|Df\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)}^2.$$

The next proposition is an obvious consequence of Assumption 2.4.

Proposition 2.9. Under Assumption 2.4 there exist a positive constants C_{Θ} such that the following estimates hold,

$$\sup_{t \in [0, T]} \|\Theta(t)\|_{W^{1,2}(\mathcal{H}, \mu)} < C_{\Theta} \quad (2.14)$$

and

$$\int_0^T \left\| \frac{\partial \Theta}{\partial t}(t) \right\|_{L^2(\mathcal{H}, \mu)}^2 dt < C_{\Theta}. \quad (2.15)$$

In the next section we provide an algorithm for the finite dimensional reduction of the optimal stopping problem (2.7).

3 The approximation scheme

The algorithm requires two separate steps. First, a Yosida approximation of the unbounded operator A by bounded operators A_{α} ; then a finite dimensional reduction of the SDE. At each step a corresponding optimal stopping problem is studied.

3.1 Yosida approximation

A natural way to deal with an unbounded linear operator is to introduce its Yosida approximation, which does not require any further assumptions. The Yosida approximation of A is defined as $A_{\alpha} := \alpha A(\alpha I - A)^{-1}$, for $\alpha > 0$ (cf. [28]). The corresponding SDE is

$$\begin{cases} dX_t^{(\alpha)x} = A_{\alpha} X_t^{(\alpha)x} dt + \sigma(X_t^{(\alpha)x}) dW_t^0, & t \in [0, S], \\ X_0^{(\alpha)x} = x, \end{cases} \quad (3.1)$$

which admits a unique strong solution, $X^{(\alpha)x}$, since A_α is a bounded linear operator. That is,

$$X_t^{(\alpha)x} = x + \int_0^t A_\alpha X_s^{(\alpha)x} ds + \int_0^t \sigma(X_s^{(\alpha)x}) dW_s^0, \quad t \in [0, S], \mathbb{P}\text{-a.s.}$$

Clearly a strong solution is also a mild solution (cf. [10]), hence $X^{(\alpha)x}$ might be equivalently interpreted as

$$X_t^{(\alpha)x} = e^{tA_\alpha} x + \int_0^t e^{(t-s)A_\alpha} \sigma(X_s^{(\alpha)x}) dW_s^0, \quad t \in [0, S], \mathbb{P}\text{-a.s.}$$

For each $\alpha > 0$, the notations $X^{(\alpha)x}$ and $X^{(\alpha)t,x}$ are analogous to those used in Section 2. The following important convergence result is proven in [10], Proposition 7.5 and it is here recalled for completeness.

Proposition 3.1. *Let X^x be the unique mild solution of equation (2.2) and $X^{(\alpha)x}$ the unique strong solution of equation (3.1). For $p \geq 1$, the following convergence holds*

$$\lim_{\alpha \rightarrow \infty} \mathbb{E} \left\{ \sup_{0 \leq t \leq S} \|X_t^{(\alpha)x} - X_t^x\|_{\mathcal{H}}^p \right\} = 0, \quad x \in \mathcal{H}.$$

We define \mathcal{U}_α to be the value function of the optimal stopping problem corresponding to $X^{(\alpha)x}$,

$$\mathcal{U}_\alpha(t, x) := \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ \Theta(\tau, X_\tau^{(\alpha)t,x}) \right\}. \quad (3.2)$$

Notice that \mathcal{U}_α satisfies (2.10) and (2.11) with the same constants. We have the convergence of \mathcal{U}_α to \mathcal{U} (cf. (2.7)) as $\alpha \rightarrow \infty$ both uniformly with respect to t and in a suitable L^p -norm.

Theorem 3.2. *The following convergence results hold.*

$$\lim_{\alpha \rightarrow \infty} \sup_{0 \leq t \leq T} |\mathcal{U}_\alpha(t, x) - \mathcal{U}(t, x)| = 0, \quad x \in \mathcal{H}, \quad (3.3)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^T \int_{\mathcal{H}} |\mathcal{U}_\alpha(t, x) - \mathcal{U}(t, x)|^p \mu(dx) dt = 0, \quad 1 \leq p < \infty, \quad (3.4)$$

for any finite measure μ on the Hilbert space \mathcal{H} .

Proof. The arguments are similar to those used in the proof of Proposition 2.7. In fact by the Lipschitz property of the gain function Θ and the time-homogeneous character of the processes we have

$$|\mathcal{U}_\alpha(t, x) - \mathcal{U}(t, x)| \leq L_{\mathcal{U}} \mathbb{E} \left\{ \sup_{0 \leq s \leq S} \|X_s^{(\alpha)x} - X_s^x\|_{\mathcal{H}} \right\}.$$

Since $L_{\mathcal{U}}$ is independent of t , the uniform convergence (3.3) follows from Proposition 3.1. To prove (3.4) it suffices to apply the dominated convergence theorem, since \mathcal{U}_α is uniformly bounded by $\overline{\Theta}$. \square

Some crucial properties of the convergence as $\alpha \rightarrow \infty$ are obtained below from standard results in Analysis.

Theorem 3.3. *If $\mathcal{U}_\alpha \in C_b([0, T] \times \mathcal{H})$ for all $\alpha > 0$, then $\mathcal{U}_\alpha \rightarrow \mathcal{U}$ as $\alpha \rightarrow \infty$, uniformly on compact subsets $[0, T] \times \mathcal{K} \subset [0, T] \times \mathcal{H}$. Moreover $\mathcal{U}(t, x) \in C_b([0, T] \times \mathcal{H})$.*

Proof. Fix $x \in \mathcal{H}$, then (3.3) implies $\mathcal{U}(\cdot, x) \in C_b([0, T]; \mathbb{R})$. For each $\alpha > 0$ define

$$F_\alpha(x) := \sup_{t \in [0, T]} |\mathcal{U}_\alpha(t, x) - \mathcal{U}(t, x)|,$$

then $F_\alpha(x) \rightarrow 0$ as $\alpha \rightarrow \infty$ by (3.3). The family $(F_\alpha)_{\alpha > 0}$ is equibounded and equi-continuous since (2.10) and (2.11) hold for both \mathcal{U}_α and \mathcal{U} , and

$$\begin{aligned} |F_\alpha(x) - F_\alpha(y)| &\leq \sup_{t \in [0, T]} |\mathcal{U}_\alpha(t, x) - \mathcal{U}_\alpha(t, y) + \mathcal{U}(t, y) - \mathcal{U}(t, x)| \\ &\leq \sup_{t \in [0, T]} |\mathcal{U}_\alpha(t, x) - \mathcal{U}_\alpha(t, y)| + \sup_{t \in [0, T]} |\mathcal{U}(t, y) - \mathcal{U}(t, x)| \leq 2L_\mathcal{U} \|x - y\|_\mathcal{H}. \end{aligned}$$

Then \mathcal{U}_α converges uniformly to \mathcal{U} , as $\alpha \rightarrow \infty$, on compact subsets $[0, T] \times \mathcal{K}$ ([13], Theorem 7.5.6); that is

$$\lim_{\alpha \rightarrow \infty} \sup_{(t, x) \in [0, T] \times \mathcal{K}} |\mathcal{U}_\alpha(t, x) - \mathcal{U}(t, x)| = 0.$$

Hence, being the uniform limit of bounded continuous functions, \mathcal{U} is continuous on any compact subset $[0, T] \times \mathcal{K}$ (cf. [13], Theorem 7.2.1).

That is enough to show the continuity of \mathcal{U} on $[0, T] \times \mathcal{H}$. In fact, fix $(t, x) \in [0, T] \times \mathcal{H}$ and let $(t_n, x_n) \in [0, T] \times \mathcal{H}$ be a sequence converging to (t, x) . Then

$$\begin{aligned} |\mathcal{U}(t_n, x_n) - \mathcal{U}(t, x)| &\leq |\mathcal{U}(t_n, x_n) - \mathcal{U}(t_n, x)| + |\mathcal{U}(t_n, x) - \mathcal{U}(t, x)| \\ &\leq L_\mathcal{U} \|x_n - x\|_\mathcal{H} + |\mathcal{U}(t_n, x) - \mathcal{U}(t, x)| \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, by (2.11) and $\mathcal{U}(\cdot, x) \in C_b([0, T]; \mathbb{R})$. \square

3.2 Finite dimensional reduction

For each $n \in \mathbb{N}$ let us consider the finite dimensional subset $\mathcal{H}^{(n)} := \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ and the orthogonal projection operator $P_n : \mathcal{H} \rightarrow \mathcal{H}^{(n)}$. We approximate the diffusion coefficients of (3.1), respectively, by $\sigma^{(n)} := (P_n \sigma) \circ P_n$ and $A_{\alpha, n} := P_n A_\alpha P_n$. Notice that $A_{\alpha, n}$ is a bounded linear operator on $\mathcal{H}^{(n)}$. We define the process $X^{(\alpha)x; n}$ as the unique strong solution of the SDE on $\mathcal{H}^{(n)}$ given by

$$\begin{cases} dX_t^{(\alpha)x; n} = A_{\alpha, n} X_t^{(\alpha)x; n} dt + \sigma^{(n)}(X_t^{(\alpha)x; n}) dW_t^0 + \epsilon_n \sum_{i=1}^n \varphi_i dW_t^i, & t \in [0, S], \\ X_0^{(\alpha)x; n} = P_n x =: x^{(n)}, \end{cases} \quad (3.5)$$

where $(\epsilon_n)_n$ is a sequence of positive numbers such that

$$\sqrt{n} \epsilon_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.6)$$

Obviously $X^{(\alpha)x; n}$ lives in the finite dimensional subspace $\mathcal{H}^{(n)}$ but it may still be seen as a solution in \mathcal{H} .

Remark 3.4. Notice that at each time $t \in [0, S]$, $X_t^{(\alpha)x; n}$ is not the projection of the process $X_t^{(\alpha)x}$ on the finite dimensional subspace. In fact, a process with that property would not be Markovian. Hence $X^{(\alpha)x; n}$ has to be considered as an auxiliary diffusion process which is used to approximate the original one.

Proposition 3.5. *It holds that*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \sup_{t \in [0, S]} \|X_t^{(\alpha)x;n} - X_t^{(\alpha)x}\|^2 \right\} = 0, \quad (3.7)$$

uniformly with respect to x on compact subsets of \mathcal{H} .

Proof. Since $X^{(\alpha)x;n}$ and $X^{(\alpha)x}$ are both strong solutions, i.e.

$$\begin{aligned} X^{(\alpha)x;n} &= P_n x + \int_0^t A_{\alpha,n} X_s^{(\alpha)x;n} ds + \int_0^t \sigma^{(n)}(X_s^{(\alpha)x;n}) dW_s^0 + \epsilon_n \sum_{i=1}^n \varphi_i W_t^i, \\ X^{(\alpha)x} &= x + \int_0^t A_\alpha X_s^{(\alpha)x} ds + \int_0^t \sigma(X_s^{(\alpha)x}) dW_s^0, \end{aligned}$$

we have

$$\begin{aligned} \|X_t^{(\alpha)x;n} - X_t^{(\alpha)x}\|_{\mathcal{H}}^2 &\leq 6 \left[\|P_n x - x\|_{\mathcal{H}}^2 + \left\| \int_0^t P_n A_\alpha (X_s^{(\alpha)x;n} - X_s^{(\alpha)x}) ds \right\|_{\mathcal{H}}^2 \right. \\ &\quad + \left\| \int_0^t (I - P_n) A_\alpha X_s^{(\alpha)x} ds \right\|_{\mathcal{H}}^2 + \left\| \int_0^t P_n [\sigma(X_s^{(\alpha)x;n}) - \sigma(X_s^{(\alpha)x})] dW_s^0 \right\|_{\mathcal{H}}^2 \\ &\quad \left. + \left\| \int_0^t (I - P_n) \sigma(X_s^{(\alpha)x}) dW_s^0 \right\|_{\mathcal{H}}^2 + \epsilon_n^2 \sum_{i=1}^n |W_t^i|^2 \right], \end{aligned}$$

where we used the fact that $A_{\alpha,n} X^{(\alpha)x;n} = P_n A_\alpha X^{(\alpha)x;n}$.

By using Hölder's inequality and by taking the supremum over $t \in [0, S]$ we obtain

$$\begin{aligned} \sup_{0 \leq t \leq S} \|X_t^{(\alpha)x;n} - X_t^{(\alpha)x}\|_{\mathcal{H}}^2 &\leq 6 \left[\|P_n x - x\|_{\mathcal{H}}^2 + S \|A_\alpha\|_L^2 \int_0^S \sup_{0 \leq u \leq s} \|X_u^{(\alpha)x;n} - X_u^{(\alpha)x}\|_{\mathcal{H}}^2 ds \right. \\ &\quad + S \int_0^S \|(I - P_n) A_\alpha X_s^{(\alpha)x}\|_{\mathcal{H}}^2 ds \\ &\quad + \sup_{0 \leq t \leq S} \left\| \int_0^t P_n [\sigma(X_s^{(\alpha)x;n}) - \sigma(X_s^{(\alpha)x})] dW_s^0 \right\|_{\mathcal{H}}^2 \\ &\quad \left. + \sup_{0 \leq t \leq S} \left\| \int_0^t (I - P_n) \sigma(X_s^{(\alpha)x}) dW_s^0 \right\|_{\mathcal{H}}^2 + \epsilon_n^2 \sum_{i=1}^n \sup_{0 \leq t \leq S} |W_t^i|^2 \right]. \end{aligned}$$

Then we pass to the expectation. Denote by $\|\cdot\|_L$ the operatorial norm of linear operators on \mathcal{H} . An estimate like (2.8) and an application of Fubini's theorem give us

$$\begin{aligned} \mathbb{E} \left\{ \sup_{0 \leq t \leq S} \|X_t^{(\alpha)x;n} - X_t^{(\alpha)x}\|_{\mathcal{H}}^2 \right\} &\leq 6 \left[\|P_n x - x\|_{\mathcal{H}}^2 + S \|A_\alpha\|_L^2 \int_0^S \mathbb{E} \left\{ \sup_{0 \leq u \leq s} \|X_u^{(\alpha)x;n} - X_u^{(\alpha)x}\|_{\mathcal{H}}^2 \right\} ds \right. \\ &\quad + S \int_0^S \mathbb{E} \left\{ \|(I - P_n) A_\alpha X_s^{(\alpha)x}\|_{\mathcal{H}}^2 \right\} ds \\ &\quad + \int_0^S \mathbb{E} \left\{ \|\sigma(X_s^{(\alpha)x;n}) - \sigma(X_s^{(\alpha)x})\|_{\mathcal{H}}^2 \right\} ds \\ &\quad \left. + \int_0^S \mathbb{E} \left\{ \|(I - P_n) \sigma(X_s^{(\alpha)x})\|_{\mathcal{H}}^2 \right\} ds + \epsilon_n^2 \sum_{i=1}^n \mathbb{E} \left\{ \sup_{0 \leq t \leq S} |W_t^i|^2 \right\} \right]. \end{aligned}$$

By Assumption 2.2 the diffusion coefficient is Lipschitz and we denote by $L_\sigma > 0$ the Lipschitz constant. Then we get

$$\begin{aligned} \mathbb{E} \left\{ \sup_{0 \leq t \leq S} \|X_t^{(\alpha)x;n} - X_t^{(\alpha)x}\|_{\mathcal{H}}^2 \right\} &\leq 6 \left[\|P_n x - x\|_{\mathcal{H}}^2 + S \|A_\alpha\|_L^2 \int_0^S \mathbb{E} \left\{ \sup_{0 \leq u \leq s} \|X_u^{(\alpha)x;n} - X_u^{(\alpha)x}\|_{\mathcal{H}}^2 \right\} ds \right. \\ &\quad + S \int_0^S \mathbb{E} \left\{ \|(I - P_n)A_\alpha X_s^{(\alpha)x}\|_{\mathcal{H}}^2 \right\} ds \\ &\quad + L_\sigma^2 \int_0^S \mathbb{E} \left\{ \sup_{0 \leq u \leq s} \|X_u^{(\alpha)x;n} - X_u^{(\alpha)x}\|_{\mathcal{H}}^2 \right\} ds \\ &\quad \left. + \int_0^S \mathbb{E} \left\{ \|(I - P_n)\sigma(X_s^{(\alpha)x})\|_{\mathcal{H}}^2 \right\} ds + \epsilon_n^2 n S \right]. \end{aligned}$$

A straightforward application of Gronwall's lemma gives

$$\begin{aligned} \mathbb{E} \left\{ \sup_{0 \leq t \leq S} \|X_t^{(\alpha)x;n} - X_t^{(\alpha)x}\|_{\mathcal{H}}^2 \right\} &\leq C_S \left\{ \|P_n x - x\|_{\mathcal{H}}^2 + \epsilon_n^2 S \right. \\ &\quad \left. + \int_0^S \mathbb{E} \left\{ \|(I - P_n)A_\alpha X_s^{(\alpha)x}\|_{\mathcal{H}}^2 + \|(I - P_n)\sigma(X_s^{(\alpha)x})\|_{\mathcal{H}}^2 \right\} ds \right\} \exp(S^2 \|A_\alpha\|_L^2 + S L_\sigma^2), \end{aligned} \quad (3.8)$$

for some positive constant C_S . The right hand side converges to zero as $n \rightarrow \infty$ by dominated convergence and the hypothesis about ϵ_n .

We want to prove that the limit is uniform on compact subsets of \mathcal{H} . For each n we define the real function $M_n(x)$ by

$$M_n(x) := \|P_n x - x\|_{\mathcal{H}}^2 + \epsilon_n^2 n S + \int_0^S \mathbb{E} \left\{ \|(I - P_n)A_\alpha X_s^{(\alpha)x}\|_{\mathcal{H}}^2 + \|(I - P_n)\sigma(X_s^{(\alpha)x})\|_{\mathcal{H}}^2 \right\} ds.$$

Notice that $M_n(x)$ is continuous in \mathcal{H} for all $n \geq 1$. In fact, for $x, y \in \mathcal{H}$, we may apply Lemma 2.6 to

$$\begin{aligned} &\left| \int_0^S \mathbb{E} \left\{ \|(I - P_n)A_\alpha X_s^{(\alpha)x}\|_{\mathcal{H}}^2 - \|(I - P_n)A_\alpha X_s^{(\alpha)y}\|_{\mathcal{H}}^2 \right\} ds \right| \\ &+ \left| \int_0^S \mathbb{E} \left\{ \|(I - P_n)\sigma(X_s^{(\alpha)x})\|_{\mathcal{H}}^2 - \|(I - P_n)\sigma(X_s^{(\alpha)y})\|_{\mathcal{H}}^2 \right\} ds \right| \\ &\leq (\|A_\alpha\|_L^2 + L_\sigma^2) S \mathbb{E} \left\{ \sup_{0 \leq u \leq S} \|X_u^{(\alpha)x} - X_u^{(\alpha)y}\|_{\mathcal{H}}^2 \right\}. \end{aligned}$$

Moreover, $M_n(x)$ decreases to zero as $n \rightarrow \infty$. Hence Dini's theorem guarantees uniform convergence on any compact subset $\mathcal{K} \subset \mathcal{H}$. \square

Remark 3.6. Notice that the previous proposition and the arguments of its proof hold for $X^{(\alpha)t,x;n}$ and $X^{(\alpha)t,x}$ as well, for any starting time $t \in [0, S]$, thanks to the time-homogeneous character of equations (3.1) and (3.5).

Denote by $I_{\{s > \tau\}}$ the indicator function of the set $\{\omega \in \Omega : s > \tau(\omega)\}$, where τ is a stopping time and define $[x]^+ := \max\{x, 0\}$. The following technical result will be needed in the next sections. Its proof is provided in the appendix.

Lemma 3.7. For every stopping time τ such that $\tau \in [t, T]$ \mathbb{P} -a.s., it holds

$$\mathbb{E} \left\{ \sup_{t \leq s \leq T} \|X_s^{(\alpha)t,x;n} - X_\tau^{(\alpha)t,x;n}\|_{\mathcal{H}}^2 I_{\{s > \tau\}} \right\} \leq C_{\alpha,n,T} (1 + \|x\|_{\mathcal{H}}^4) \mathbb{E} \{ [T - \tau]^+ \}^{\frac{1}{2}} \quad (3.9)$$

for a positive constant $C_{\alpha,n,T}$ depending only on α, n, T .

For $n \geq 1$ define $\Theta^{(n)} : [0, S] \times \mathcal{H} \rightarrow \mathbb{R}$ by

$$\Theta^{(n)}(t, x) := \Theta(t, P_n x) = \Theta(t, x^{(n)}) \quad (3.10)$$

(cf. (3.5)). This function maps \mathcal{H} into $\mathcal{H}^{(n)}$ and it will be largely used in Section 5. Of course, $P_n x^{(n)} = x^{(n)}$, hence $\Theta^{(n)}(t, \cdot) = \Theta(t, \cdot)$ on $\mathcal{H}^{(n)}$. However, in what follows it is convenient to use the notation $\Theta^{(n)}$ since this is a gain function on $\mathcal{H}^{(n)}$ and it will occur in the variational formulation of a finite dimensional optimal stopping problem approximating (3.2). It is not hard to see that from (2.5) and Dini's Theorem follows

$$\lim_{n \rightarrow \infty} \sup_{(t,x) \in [0,T] \times \mathcal{K}} |\Theta^{(n)}(t, x) - \Theta(t, x)| = 0, \quad \text{for every compact } \mathcal{K} \subset \mathcal{H}. \quad (3.11)$$

Remark 3.8. *There is an isomorphism $\mathcal{I}_n : (\mathcal{H}^{(n)}, \|\cdot\|_{\mathcal{H}}) \rightarrow (\mathbb{R}^n, \|\cdot\|_{\mathbb{R}^n})$, in fact for any $x \in \mathcal{H}^{(n)}$ we may define $x_i := \langle x, \varphi_i \rangle_{\mathcal{H}}$, $i = 1, 2, \dots, n$ and $\mathcal{I}_n x := (x_1, \dots, x_n)$.*

Let $\mathcal{U}_\alpha^{(n)}$ be the value function of the optimal stopping problem

$$\mathcal{U}_\alpha^{(n)}(t, x^{(n)}) := \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ \Theta^{(n)}(\tau, X_\tau^{(\alpha)t, x; n}) \right\}. \quad (3.12)$$

Obviously $\mathcal{U}_\alpha^{(n)}$ may also be seen as a function defined on $[0, T] \times \mathbb{R}^n$. Again, as for \mathcal{U}_α , we point out that $\mathcal{U}_\alpha^{(n)}$ satisfies (2.10) and (2.11) with the same constants. The value function $\mathcal{U}_\alpha^{(n)}$ converges to \mathcal{U}_α of (3.2) as $n \rightarrow \infty$. In fact results similar to Theorem 3.2 and Theorem 3.3 hold.

Theorem 3.9. *The following convergence results hold,*

$$\lim_{n \rightarrow \infty} \sup_{(t,x) \in [0,T] \times \mathcal{K}} |\mathcal{U}_\alpha^{(n)}(t, x^{(n)}) - \mathcal{U}_\alpha(t, x)| = 0, \quad \mathcal{K} \subset \mathcal{H}, \quad \mathcal{K} \text{ compact}, \quad (3.13)$$

i.e. the convergence is uniform on any compact subset $[0, T] \times \mathcal{K}$, and

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\mathcal{H}} |\mathcal{U}_\alpha^{(n)}(t, x^{(n)}) - \mathcal{U}_\alpha(t, x)|^p \mu(dx) dt = 0, \quad 1 \leq p < \infty, \quad (3.14)$$

for any finite measure μ on the Hilbert space \mathcal{H} .

Proof. The proof follows along the same lines as the proof of Theorem 3.2 since $\Theta^{(n)}(t, X_s^{(\alpha)t, x; n}) = \Theta(t, X_s^{(\alpha)t, x; n})$, $s \geq t$. Then (3.13) follows from the uniform convergence in Proposition 3.5, and (3.14) follows from dominated convergence. \square

As a consequence we have

Theorem 3.10. *If $\mathcal{U}_\alpha^{(n)} \in C_b([0, T] \times \mathcal{H}^{(n)})$ for all $n \geq 1$, then $\mathcal{U}_\alpha \in C_b([0, T] \times \mathcal{H})$.*

Proof. Recall that $(\mathcal{U}_\alpha^{(n)}(t, x^{(n)}))_n$ is uniformly bounded (cf. Proposition 2.7) and (3.13) holds. Hence [13], Theorem 7.2.1 guarantees the continuity of \mathcal{U}_α on $[0, T] \times \mathcal{K}$. Arguments as in Theorem 3.3 provide the continuity on $[0, T] \times \mathcal{H}$. \square

Later in the paper we will prove that $\mathcal{U}_\alpha^{(n)}$ is indeed continuous.

4 Finite dimensional variational inequality

In Appendix B a variational formulation of the finite dimensional, optimal stopping problem (3.12) is established in the case of bounded domains by a suitable application of the theory developed in [3] by Bensoussan and Lions. Also, in [3] such result is generalized to unbounded domains by means of weighted Sobolev spaces. Here we deal with unbounded domains by adapting the arguments in [3] to specific weighted Sobolev spaces that will allow an extension to the infinite-dimensional case.

We introduce some notation. Recall the notation $L^p(\mathbb{R}^n, \mu_n)$ of Remark 2.8.

Definition 4.1. For $1 \leq p < \infty$ set

$$\mathcal{V}_n^p := \{v : v \in L^p(\mathbb{R}^n, \mu_n) \text{ and } Dv \in L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)\} \quad (4.1)$$

and endow it with the norm

$$\|v\|_{p,n} := \|v\|_{L^p(\mathbb{R}^n, \mu_n)} + \|Dv\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)}. \quad (4.2)$$

Then $(\mathcal{V}_n^p, \|\cdot\|_{p,n})$ is a separable Banach space.

Denote by $(\cdot, \cdot)_{\mu_n}$ the scalar product in $L^2(\mathbb{R}^n, \mu_n)$ and, for $u, w \in \mathcal{V}_n^p$, define the bilinear form associated to the operator $\mathcal{L}_{\alpha,n}$ (cf. (B-3)),

$$\begin{aligned} a_{\mu}^{(\alpha,n)}(u, w) &:= - \int_{\mathbb{R}^n} \mathcal{L}_{\alpha,n} u w \mu_n(dx) \\ &= \frac{1}{2} \sum_{i,j=1}^n \left[\int_{\mathbb{R}^n} ([\sigma^{(n)} \sigma^{(n)*}]_{i,j} + \epsilon_n^2 \delta_{i,j}) \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} \mu_n(dx) + \int_{\mathbb{R}^n} \frac{\partial [\sigma^{(n)} \sigma^{(n)*}]_{i,j}}{\partial x_j} \frac{\partial u}{\partial x_i} w \mu_n(dx) \right. \\ &\quad \left. - \int_{\mathbb{R}^n} \left([\sigma^{(n)} \sigma^{(n)*}]_{i,j} \frac{x_j}{\lambda_j} + 2x_j \langle A_{\alpha} \varphi_j, \varphi_i \rangle + \epsilon_n^2 \delta_{i,j} \frac{x_j}{\lambda_j} \right) \frac{\partial u}{\partial x_i} w \mu_n(dx) \right]. \end{aligned} \quad (4.3)$$

By looking at (B-4) one notices that

$$\frac{\partial}{\partial x_j} [\sigma^{(n)} \sigma^{(n)*}(x)]_{i,j} = \langle D\sigma^{(n)}(x) \varphi_j, \varphi_i \rangle_{\mathcal{H}} \langle \sigma^{(n)}(x), \varphi_j \rangle_{\mathcal{H}} + \langle D\sigma^{(n)}(x) \varphi_j, \varphi_j \rangle_{\mathcal{H}} \langle \sigma^{(n)}(x), \varphi_i \rangle_{\mathcal{H}}. \quad (4.4)$$

The isometry $\mathcal{H}^{(n)} \sim \mathbb{R}^n$ and (4.4) allow to rewrite the bilinear form (4.3) as

$$\begin{aligned} a_{\mu}^{(\alpha,n)}(u, w) &:= \frac{1}{2} \int_{\mathbb{R}^n} \left(\langle \sigma^{(n)} \sigma^{(n)*} Du, Dw \rangle_{\mathcal{H}} + \epsilon_n^2 \langle Du, Dw \rangle_{\mathcal{H}} \right) \mu_n(dx) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} \left(\text{Tr}[D\sigma^{(n)}]_{\mathcal{H}} \langle \sigma^{(n)}, Du \rangle_{\mathcal{H}} + \langle D\sigma^{(n)} \cdot \sigma^{(n)}, Du \rangle_{\mathcal{H}} \right) w \mu_n(dx) \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^n} \langle 2A_{\alpha,n} x + \sigma^{(n)} \sigma^{(n)*} Q_n^{-1} x + \epsilon_n^2 Q_n^{-1} x, Du \rangle_{\mathcal{H}} w \mu_n(dx) \end{aligned} \quad (4.5)$$

where $Q_n := P_n Q P_n$ and $(D\sigma^{(n)} \cdot \sigma^{(n)})_i := \sum_{j=1}^n (D\sigma^{(n)})_{i,j} \sigma_j^{(n)}$, $i = 1, \dots, n$. Now the continuity of the bilinear form (4.5) in \mathcal{V}_n^p follows from the next result.

Theorem 4.2. For every $2 < p < \infty$ there exists a constant $C_{\mu,\gamma,p} > 0$, depending on μ , p and the bounds of γ in Assumption 2.2, such that

$$\int_0^T |a_{\mu}^{(\alpha,n)}(u(t), w(t))| dt \leq C_{\mu,\gamma,p} \left(\int_0^T \|u(t)\|_{p,n}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \|w(t)\|_{p,n}^2 dt \right)^{\frac{1}{2}} \quad (4.6)$$

for all $u, w \in L^2(0, T; \mathcal{V}_n^p)$.

Proof. Thanks to Assumptions 2.2 and 2.3 the estimate is straightforward for the terms in (4.5) except those involving Q_n^{-1} and $A_{\alpha,n}$. For those we look at

$$(I) := \left| \int_{\mathbb{R}^n} \langle A_{\alpha,n} x, Du \rangle_{\mathcal{H}} w \mu_n(dx) \right|, \quad (4.7)$$

$$(II) := \left| \frac{1}{2} \int_{\mathbb{R}^n} \langle \sigma^{(n)} \sigma^{(n)*} Q_n^{-1} x, Du \rangle_{\mathcal{H}} w \mu_n(dx) \right|, \quad (4.8)$$

$$(III) := \epsilon_n^2 \frac{1}{2} \left| \int_{\mathbb{R}^n} \langle Q_n^{-1} x, Du \rangle_{\mathcal{H}} w \mu_n(dx) \right|. \quad (4.9)$$

For (I) it is not hard to see that

$$(I) \leq \sum_{j=1}^n \|A_{\alpha} \varphi_j\|_{\mathcal{H}} \int_{\mathbb{R}^n} \|Du\|_{\mathcal{H}} |x_j| |w| \mu_n(dx).$$

Take $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and use twice Hölder's inequality to obtain

$$(I) \leq \|Du\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)} \|w\|_{L^{2p}(\mathbb{R}^n, \mu_n)} \sum_{j=1}^n \|A_{\alpha} \varphi_j\|_{\mathcal{H}} \left(\int_{\mathbb{R}^n} |x_j|^{2q} \mu_n(dx) \right)^{\frac{1}{2q}}. \quad (4.10)$$

Recall that $\|A_{\alpha} \varphi_j\|_{\mathcal{H}} \leq M \|A \varphi_j\|_{\mathcal{H}}$ for a suitable constant $M > 0$ (cf. [28]); evaluate the integral with respect to the Gaussian measure in (4.10) and extend the sum to infinitely many terms to obtain

$$(I) \leq C_{p,M} \|Du\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)} \|w\|_{L^{2p}(\mathbb{R}^n, \mu_n)} \sum_{j=1}^{\infty} \|A \varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j}, \quad (4.11)$$

where $C_{p,M} > 0$ is a constant depending only on p and M . Notice that the right hand side of (4.11) is well defined thanks to Assumption 2.3.

Similar estimates hold for (II) and (III) of (4.8) and (4.9) by again using Hölder's inequality twice as in (4.10) and by recalling Assumption 2.2 (1); that is

$$(II) \leq C_{p,\gamma} \|Du\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)} \|w\|_{L^{2p}(\mathbb{R}^n, \mu_n)} \sum_{j=1}^{\infty} \sqrt{\lambda_j}, \quad (4.12)$$

$$(III) \leq C_{p,\gamma} \epsilon_n^2 \frac{1}{2} \|Du\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)} \|w\|_{L^p(\mathbb{R}^n, \mu_n)} \sum_{i=1}^n \frac{1}{\sqrt{\lambda_i}}, \quad (4.13)$$

for a suitable constant $C_{p,\gamma} > 0$ depending on p and on the bounds on γ in Assumption 2.2. Recall that ϵ_n satisfies $\sqrt{n} \epsilon_n \rightarrow 0$ (cf. (3.6)); in particular by taking $\epsilon_n \leq 1/n \min_{i=1,\dots,n} (\lambda_i)^{1/4}$, (3.6) is fulfilled and the right hand side of (4.13) remains bounded. In fact

$$(III) \leq \frac{1}{2n} C_{p,\gamma} \|Du\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)} \|w\|_{L^p(\mathbb{R}^n, \mu_n)}. \quad (4.14)$$

Now (4.11), (4.12) and (4.14) imply (4.6). \square

In the spirit of [3], Chapter 3, Section 1.11, take $w_R \in \mathcal{K}_{n,R}$ (cf. (B-7)) and recall that $u_{\alpha,R}^{(n)}$ is the unique solution of (B-14). Define $\tilde{w}_R \in \mathcal{K}_{n,R}$ by

$$\tilde{w}_R(x^{(n)}) - u_{\alpha,R}^{(n)}(x^{(n)}) := \frac{1}{\sqrt{(2\pi)^n \lambda_1 \lambda_2 \cdots \lambda_n}} \exp \left(- \sum_{i=1}^n \frac{x_i^2}{\lambda_i} \right) (w_R(x^{(n)}) - u_{\alpha,R}^{(n)}(x^{(n)})). \quad (4.15)$$

Take $w = \tilde{w}_R$ in (B-14) and use (4.15) to obtain

$$\int_0^T \left[-\left(\frac{\partial w_R}{\partial t}, w_R - u_{\alpha,R}^{(n)}\right)_{\mu_n} + a_{\mu}^{(\alpha,n)}(u_{\alpha,R}^{(n)}, w_R - u_{\alpha,R}^{(n)}) - (f_{\alpha,n}, w_R - u_{\alpha,R}^{(n)})_{\mu_n} \right] dt + \frac{1}{2} \|w_R(T)\|_{L^2(\mathbb{R}^n, \mu_n)}^2 \geq 0. \quad (4.16)$$

It is not hard to see that $\|f_{\alpha,n}\|_{L^2(0,T;L^p(\mathcal{H},\mu))} \leq C_{\alpha,p}$ with $C_{\alpha,p} > 0$ depending on α and $2 \leq p < \infty$ only (cf. (B-9) and Assumptions 2.2 and 2.4). It follows that (4.16) is well defined for all $n \in \mathbb{N}$ and $R > 0$.

For every $2 < p < \infty$, denote by $\mathcal{K}_{n,\mu}^p$ the closed convex set

$$\mathcal{K}_{n,\mu}^p := \{w : w \in L^2(0,T; \mathcal{V}_n^p), \frac{\partial w}{\partial t} \in L^2(0,T; L^2(\mathbb{R}^n, \mu_n)), w \geq 0 \text{ a.e. in } (0,T) \times \mathbb{R}^n\}. \quad (4.17)$$

We are now ready to extend Theorem B.4 to the unbounded case, i.e. to \mathbb{R}^n . Recall (3.10) and the optimal stopping problem (3.12) and set

$$u_{\alpha}^{(n)} := \mathcal{U}_{\alpha}^{(n)} - \Theta^{(n)}. \quad (4.18)$$

Theorem 4.3. *For every $2 < p < \infty$ the function $u_{\alpha}^{(n)}$ is a solution of the weak variational problem on \mathbb{R}^n*

$$\left\{ \begin{array}{l} u \in L^2(0,T; \mathcal{V}_n^p); \quad u(T, x^{(n)}) = 0, x^{(n)} \in \mathbb{R}^n; \quad u(t, x^{(n)}) \geq 0, (t, x^{(n)}) \in [0,T] \times \mathbb{R}^n; \\ \int_0^T \left[-\left(\frac{\partial w}{\partial t}, w - u\right)_{\mu_n} + a_{\mu}^{(\alpha,n)}(u, w - u) - (f_{\alpha,n}, w - u)_{\mu_n} \right] dt + \frac{1}{2} \|w(T)\|_{L^2(\mathbb{R}^n, \mu_n)}^2 \geq 0 \end{array} \right. \quad (4.19)$$

for all $w \in \mathcal{K}_{n,\mu}^p$,

and $u_{\alpha}^{(n)} \in C([0,T] \times \mathbb{R}^n)$.

Moreover, the optimal stopping time for $\mathcal{U}_{\alpha}^{(n)}$ of (3.12) is

$$\tau_{\alpha,n}^*(t, x) := \inf\{s \geq t : \mathcal{U}_{\alpha}^{(n)}(s, X_s^{(\alpha)t,x;n}) = \Theta^{(n)}(s, X_s^{(\alpha)t,x;n})\} \wedge T. \quad (4.20)$$

Proof. Observe that, by a slight generalization of the arguments in [1], Theorem 3.22, on cut-off functions, for each $w \in \mathcal{K}_{n,\mu}^p$ there exists a family $(w_R)_{R>0} \subset \mathcal{K}_{n,\mu}^p$ such that $w_R|_{\partial\mathcal{O}_R} = 0$ and

$$\int_0^T \left\| \frac{\partial w_R}{\partial t} - \frac{\partial w}{\partial t} \right\|_{L^2(\mathbb{R}^n, \mu_n)}^2 dt + \int_0^T \|w_R - w\|_{n,p}^2 dt \rightarrow 0, \quad \text{as } R \rightarrow \infty. \quad (4.21)$$

Moreover $w_R \in \mathcal{K}_{n,R}$ of (B-7). Write the inequality (4.16) as

$$\begin{aligned} - \int_0^T \left(\frac{\partial w_R}{\partial t}, w_R - u_{\alpha,R}^{(n)}\right)_{\mu_n} dt + \int_0^T a_{\mu}^{(\alpha,n)}(u_{\alpha,R}^{(n)}, w_R) dt + \frac{1}{2} \|w_R(T)\|_{L^2(\mathbb{R}^n, \mu_n)}^2 \\ \geq \int_0^T (f_{\alpha,n}, w_R - u_{\alpha,R}^{(n)})_{\mu_n} dt + \int_0^T a_{\mu}^{(\alpha,n)}(u_{\alpha,R}^{(n)}, u_{\alpha,R}^{(n)}) dt \end{aligned} \quad (4.22)$$

for $R > 0$. Take $(u_{\alpha,R_i}^{(n)})_{i \in \mathbb{N}}$ and R_i as in Corollary B.6. By using [6], Proposition 3.5 and (4.21) pass to the limit as $i \rightarrow \infty$ to obtain

$$\int_0^T \left(\frac{\partial w_{R_i}}{\partial t}, w_{R_i} - u_{\alpha,R_i}^{(n)}\right)_{\mu_n} dt \rightarrow \int_0^T \left(\frac{\partial w}{\partial t}, w - u_{\alpha}^{(n)}\right)_{\mu_n} dt, \quad (4.23)$$

$$\int_0^T (f_{\alpha,n}, w_{R_i} - u_{\alpha,R_i}^{(n)})_{\mu_n} dt \rightarrow \int_0^T (f_{\alpha,n}, w - u_{\alpha}^{(n)})_{\mu_n} dt, \quad (4.24)$$

$$\int_0^T a_{\mu}^{(\alpha,n)}(u_{\alpha,R_i}^{(n)}, w_{R_i}) dt \rightarrow \int_0^T a_{\mu}^{(\alpha,n)}(u_{\alpha}^{(n)}, w) dt. \quad (4.25)$$

Moreover, by considering a subsequence (if needed) again denoted by $(w_{R_i})_{i \in \mathbb{N}}$, pointwise convergence is also obtained,

$$\|w_{R_i}(T)\|_{L^2(\mathbb{R}^n, \mu_n)}^2 \rightarrow \|w(T)\|_{L^2(\mathbb{R}^n, \mu_n)}^2. \quad (4.26)$$

To analyze the last term on the right hand side of (4.22), write it as

$$\begin{aligned} \int_0^T a_\mu^{(\alpha, n)}(u_{\alpha, R_i}^{(n)}, u_{\alpha, R_i}^{(n)}) dt &= \int_0^T a_\mu^{(\alpha, n)}(u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}, u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}) dt \\ &\quad + \int_0^T a_\mu^{(\alpha, n)}(u_\alpha^{(n)}, u_{\alpha, R_i}^{(n)}) dt + \int_0^T a_\mu^{(\alpha, n)}(u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}, u_\alpha^{(n)}) dt. \end{aligned} \quad (4.27)$$

As above the second and third terms on the right-hand side of (4.27) give

$$\lim_{i \rightarrow \infty} \int_0^T a_\mu^{(\alpha, n)}(u_\alpha^{(n)}, u_{\alpha, R_i}^{(n)}) dt = \int_0^T a_\mu^{(\alpha, n)}(u_\alpha^{(n)}, u_\alpha^{(n)}) dt, \quad (4.28)$$

$$\lim_{i \rightarrow \infty} \int_0^T a_\mu^{(\alpha, n)}(u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}, u_\alpha^{(n)}) dt = 0. \quad (4.29)$$

Also, arguments similar to those in the proof of Theorem 4.2 give

$$\begin{aligned} \int_0^T a_\mu^{(\alpha, n)}(u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}, u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}) dt \\ \geq -C_p \left\| Du_{\alpha, R_i}^{(n)} - Du_\alpha^{(n)} \right\|_{L^2(0, T; L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n))} \left\| u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)} \right\|_{L^2(0, T; L^p(\mathbb{R}^n, \mu_n))}, \end{aligned} \quad (4.30)$$

where $2 < p < \infty$ and $C_p > 0$ is a suitable constant independent of i , α and n . Lower semi-continuity of weak convergence and (B-18) give

$$\|u_\alpha^{(n)}\|_{L^2(0, T; W^{1,2}(\mathbb{R}^n, \mu_n))} \leq \liminf_{i \rightarrow \infty} \|u_{\alpha, R_i}^{(n)}\|_{L^2(0, T; W^{1,2}(\mathbb{R}^n, \mu_n))} \leq \sqrt{C_U T};$$

therefore there exists $\Lambda_p > 0$ such that (4.30) implies

$$\int_0^T a_\mu^{(\alpha, n)}(u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}, u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}) dt \geq -\Lambda_p \left\| u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)} \right\|_{L^2(0, T; L^p(\mathbb{R}^n, \mu_n))} \quad (4.31)$$

and now Corollary B.6 gives

$$\lim_{i \rightarrow \infty} \int_0^T a_\mu^{(\alpha, n)}(u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}, u_{\alpha, R_i}^{(n)} - u_\alpha^{(n)}) dt \geq 0. \quad (4.32)$$

Hence (4.28), (4.29) and (4.32) imply

$$\lim_{i \rightarrow \infty} \int_0^T a_\mu^{(\alpha, n)}(u_{\alpha, R_i}^{(n)}, u_{\alpha, R_i}^{(n)}) dt \geq \int_0^T a_\mu^{(\alpha, n)}(u_\alpha^{(n)}, u_\alpha^{(n)}) dt, \quad (4.33)$$

which together with (4.23), (4.24), (4.25) and (4.33), provides the convergence of (4.22) to (4.19) as $i \rightarrow \infty$. The continuity of $u_\alpha^{(n)}$ follows from Proposition B.1.

The proof of the optimality of $\tau_{\alpha, n}^*(t, x)$ is a simpler version of the proofs of Lemma 5.6 and Theorem 5.8 below, hence it is only outlined here. In fact, for any initial data (t, x) one has

$$\lim_{R \rightarrow \infty} \tau_{\alpha, n, R}^*(t, x) \wedge \tau_{\alpha, n}^*(t, x) = \tau_{\alpha, n}^*(t, x) \quad \mathbb{P} - \text{a.s.} \quad (4.34)$$

by an extension of [3], Chapter 3, Section 3, Theorem 3.7 and by using Proposition B.1. Also, since $\tau_{\alpha,n,R}^*$ is optimal for $\mathcal{U}_{\alpha,R}^{(n)}$ and $\tau_{\alpha,n,R}^* \wedge \tau_{\alpha,n}^* \leq \tau_{\alpha,n,R}^*$ \mathbb{P} -a.s., it follows from (B-13) that

$$\mathcal{U}_{\alpha,R}^{(n)}(t, x^{(n)}) = \mathbb{E} \left\{ \mathcal{U}_{\alpha,R}^{(n)}(\tau_{\alpha,n,R}^* \wedge \tau_{\alpha,n}^*, X_{\tau_{\alpha,n,R}^* \wedge \tau_{\alpha,n}^*}^{(\alpha)t,x;n}) \right\}. \quad (4.35)$$

Therefore in the limits as $R \rightarrow \infty$ in (4.35), Proposition B.1, the continuity of $\mathcal{U}_{\alpha}^{(n)}$ and (4.34) provide

$$\mathcal{U}_{\alpha}^{(n)}(t, x^{(n)}) = \mathbb{E} \left\{ \mathcal{U}_{\alpha}^{(n)}(\tau_{\alpha,n}^*, X_{\tau_{\alpha,n}^*}^{(\alpha)t,x;n}) \right\} = \mathbb{E} \left\{ \Theta^{(n)}(\tau_{\alpha,n}^*, X_{\tau_{\alpha,n}^*}^{(\alpha)t,x;n}) \right\}. \quad (4.36)$$

Hence $\tau_{\alpha,n}^*$ is optimal. \square

Remark 4.4. Notice that the arguments that provide (4.34) also give

$$\lim_{R \rightarrow \infty} \tau_{\alpha,n,R}^* \wedge \tau_{\alpha,n}^* \wedge \sigma = \tau_{\alpha,n}^* \wedge \sigma \quad \mathbb{P}\text{-a.s.} \quad (4.37)$$

for any stopping time σ . Therefore it holds that,

$$\mathcal{U}_{\alpha}^{(n)}(t, x^{(n)}) = \mathbb{E} \left\{ \mathcal{U}_{\alpha}^{(n)}(\sigma, X_{\sigma}^{(\alpha)t,x;n}) \right\} \quad (4.38)$$

for $\sigma \leq \tau_{\alpha,n}^*$, \mathbb{P} -a.s.

Remark 4.5. It is not clear whether a coerciveness condition similar to (B-6) holds for $a_{\mu}^{(\alpha,n)}$. That would suffice for the uniqueness of the solution of (4.19). On the other hand, a condition similar to (B-6) does not hold in the limit as $n \rightarrow \infty$ since the diffusion becomes degenerate and the uniform ellipticity of $\mathcal{L}_{\alpha,n}$ (due to the non-degeneracy of the diffusion $X^{(\alpha)t,x;n}$) is not preserved. Also, the arguments used in [25] and [26] to prove uniqueness in a finite-dimensional, degenerate case cannot be extended to our dynamics (2.2). Instead uniqueness may be recovered in the infinite-dimensional setting (cf. [2] and [37]) when μ is invariant or at least excessive for the semigroup generated by the infinitesimal generator \mathcal{L} of the diffusion.

5 Infinite dimensional variational inequality

5.1 The variational inequality

For $1 \leq p < \infty$ define the infinite-dimensional counterpart of \mathcal{V}_n^p (cf. (4.1))

$$\mathcal{V}^p := \{v : v \in L^p(\mathcal{H}, \mu) \text{ and } Dv \in L^2(\mathcal{H}, \mu; \mathcal{H})\}. \quad (5.1)$$

Endow \mathcal{V}^p with the norm

$$\|v\|_p := \|v\|_{L^p(\mathcal{H}, \mu)} + \|Dv\|_{L^2(\mathcal{H}, \mu; \mathcal{H})}, \quad (5.2)$$

so to obtain a separable Banach space. Notice that $\mathcal{V}_n^p \subset \mathcal{V}^p$ by Remark 2.8 and for $u, w \in L^2(0, T; \mathcal{V}^p)$

$$\int_0^T |a_{\mu}^{(\alpha,n)}(u(t), w(t))| dt \leq C_{\mu, \gamma, p} \left(\int_0^T \|u(t)\|_p^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \|w(t)\|_p^2 dt \right)^{\frac{1}{2}}, \quad (5.3)$$

by (4.6). Recall the value function \mathcal{U}_{α} of the optimal stopping problem (3.2). The next proposition follows from Theorem 3.9 and Proposition B.5.

Proposition 5.1. *The function $u_\alpha^{(n)}$ converges to u_α , weakly in $L^2(0, T; W^{1,2}(\mathcal{H}, \mu))$ and strongly in $L^2(0, T; L^p(\mathcal{H}, \mu))$, $1 \leq p < \infty$. Moreover u_α is identified as $u_\alpha := \mathcal{U}_\alpha - \Theta$.*

The infinitesimal generator of $X^{(\alpha)}$ (cf. (3.1)) is

$$\mathcal{L}_\alpha g(x) = \frac{1}{2} \text{Tr} [\sigma \sigma^*(x) D^2 g(x)] + \langle A_\alpha x, Dg(x) \rangle \quad \text{for } g \in C_b^2(\mathcal{H}). \quad (5.4)$$

The bilinear form associated to (5.4) is infinite-dimensional counterpart of (4.5) and is given by

$$\begin{aligned} a_\mu^{(\alpha)}(u, w) := & \frac{1}{2} \int_{\mathcal{H}} \langle \sigma \sigma^* Du, Dw \rangle_{\mathcal{H}} \mu(dx) + \frac{1}{2} \int_{\mathcal{H}} \left(\text{Tr}[D\sigma]_{\mathcal{H}} \langle \sigma, Du \rangle_{\mathcal{H}} + \langle D\sigma \cdot \sigma, Du \rangle_{\mathcal{H}} \right) w \mu(dx) \\ & - \frac{1}{2} \int_{\mathcal{H}} \langle \sigma \sigma^* Q^{-1} x + 2A_\alpha x, Du \rangle_{\mathcal{H}} w \mu(dx), \quad u, w \in L^2(0, T; \mathcal{V}^p). \end{aligned} \quad (5.5)$$

Let $w \in L^2(0, T; W^{1,2}(\mathcal{H}, \mu))$ and $(w_n)_{n \in \mathbb{N}} \subset L^2(0, T; W^{1,2}(\mathcal{H}, \mu))$ be such that $w_n \rightarrow w$. Define $\mathcal{T}_{\alpha, w}$ and the sequence $(\mathcal{T}_{\alpha, w}^n)_{n \in \mathbb{N}} \subset L^2(0, T; \mathcal{V}^p)^*$ by

$$\mathcal{T}_{\alpha, w}(u) := \int_0^T a_\mu^{(\alpha)}(u, w) dt \quad \text{and} \quad (5.6)$$

$$\mathcal{T}_{\alpha, w}^n(u) := \int_0^T a_\mu^{(\alpha, n)}(u, w_n) dt, \quad u \in L^2(0, T; \mathcal{V}^p). \quad (5.7)$$

Tedious but straightforward calculations give

$$\lim_{n \rightarrow \infty} \|\mathcal{T}_{\alpha, w}^n - \mathcal{T}_{\alpha, w}\|_{L^2(0, T; \mathcal{V}^p)^*} = 0. \quad (5.8)$$

Recall $f_{\alpha, n}$ of (B-9) and set

$$f_\alpha := \frac{\partial \Theta}{\partial t} + \mathcal{L}_\alpha \Theta. \quad (5.9)$$

From Assumptions 2.2 and 2.4, dominated convergence theorem and (B-9) follows

$$\lim_{n \rightarrow \infty} \int_0^T \|f_{\alpha, n} - f_\alpha\|_{L^p(\mathcal{H}, \mu)}^2 dt = 0, \quad 1 \leq p < \infty. \quad (5.10)$$

Recall $\mathcal{K}_{n, \mu}^p$ of (4.17) and, for $2 < p < \infty$ define the closed, convex set \mathcal{K}_μ^p

$$\mathcal{K}_\mu^p := \left\{ w : w \in L^2(0, T; \mathcal{V}^p), \frac{\partial w}{\partial t} \in L^2(0, T; L^2(\mathcal{H}, \mu)), w \geq 0 \text{ } \lambda \times \mu\text{-a.e. in } (0, T) \times \mathcal{H} \right\}, \quad (5.11)$$

where λ is the Lebesgue measure on $[0, T]$. Notice that $\mathcal{K}_\mu^p \subset C([0, T]; L^2(\mathcal{H}, \mu))$ (cf. [15], Section 5.9.2). Denote by $\|\cdot\|_{W^{1,2}}$ the norm of $W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu)$.

Theorem 5.2. *Let $w \in \mathcal{K}_\mu^p$. Then there exists in $W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu)$ a double-indexed sequence $(w_{k, n})_{k, n \in \mathbb{N}}$ such that*

$$\text{for } k \text{ fixed, } w_{k, n} \in \mathcal{K}_{m, \mu}^p \text{ for all } m \geq n \text{ and } 2 < p < \infty.$$

Moreover, taking the limits without interchanging the order

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} w_{k, n} = w \quad (5.12)$$

weakly in $W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu)$ and strongly in $L^2(0, T; L^p(\mathcal{H}, \mu))$.

Proof. Since $D(A^*)$ is dense in \mathcal{H} it is not hard to prove that the set

$$\mathcal{E}_A([0, T] \times \mathcal{H}) := \text{span} \left\{ \mathcal{R}e(\varphi_{k,h}), \mathcal{I}m(\varphi_{k,h}), \varphi_{k,h}(t, x) = e^{ikt+i\langle h, x \rangle_{\mathcal{H}}}, (k, h) \in \mathbb{N} \times D(A^*) \right\} \quad (5.13)$$

is dense¹ in both $W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu)$ and $L^2(0, T; L^p(\mathcal{H}, \mu))$ (cf. [9], Chapter 10 and [11], Chapter 9). Hence for $w \in \mathcal{K}_\mu^p$ there exists a sequence $(\phi^{(k)})_{k \in \mathbb{N}} \subset \mathcal{E}_A([0, T] \times \mathcal{H})$ such that

$$\lim_{k \rightarrow \infty} \int_0^T \left\| \phi^{(k)}(t) - w(t) \right\|_{\mathcal{V}^p}^2 dt = 0, \quad (5.14)$$

$$\lim_{k \rightarrow \infty} \int_0^T \left\| \frac{\partial \phi^{(k)}}{\partial t}(t) - \frac{\partial w}{\partial t}(t) \right\|_{L^2(\mathcal{H}, \mu)}^2 dt = 0,$$

Moreover there exists a subsequence (still denoted by $(\phi^{(k)})_{k \in \mathbb{N}}$) such that the convergence holds λ -a.e. $t \in [0, T]$.

Take $k \in \mathbb{N}$ arbitrary but fixed, then $\phi^{(k)}$ is of the form

$$\phi^{(k)}(t, x) = \sum_{m=0}^{N_k} [a_m \cos(mt + \langle h_m, x \rangle_{\mathcal{H}}) + b_m \sin(mt + \langle h_m, x \rangle_{\mathcal{H}})],$$

where $N_k \in \mathbb{N}$, $a, b \in \mathbb{R}^{N_k}$ and $(h_m)_{m \in \mathbb{N}} \subset \mathcal{H}$. Recall the projection P_n , set $\phi_n^{(k)}(t, x) := \phi^{(k)}(t, P_n x)$ for $n \in \mathbb{N}$, let $|\cdot|_k$ be the Euclidean norm in \mathbb{R}^{N_k} and define $\|h\|_k^2 := \sum_{m=0}^{N_k} \|h_m\|_{\mathcal{H}}^2$. There exist constants $C(k, |a|_k, |b|_k) > 0$ and $C(k, |a|_k, |b|_k, \|h\|_k) > 0$ independent of n and such that

$$\sup_{(t,x) \in [0,T] \times \mathcal{H}} |\phi_n^{(k)}(t, x)| \leq C(k, |a|_k, |b|_k), \quad (5.15)$$

$$\sup_{(t,x) \in [0,T] \times \mathcal{H}} \left[\left| \frac{\partial \phi_n^{(k)}}{\partial t}(t, x) \right| + \|D\phi_n^{(k)}(t, x)\|_{\mathcal{H}} \right] \leq C(k, |a|_k, |b|_k, \|h\|_k). \quad (5.16)$$

Dominated convergence, continuity of $\phi^{(k)}$ and of its derivatives give

$$\lim_{n \rightarrow \infty} \left(\int_0^T \left\| \phi_n^{(k)}(t) - \phi^{(k)}(t) \right\|_{\mathcal{V}^p}^2 dt + \int_0^T \left\| \frac{\partial \phi_n^{(k)}}{\partial t}(t) - \frac{\partial \phi^{(k)}}{\partial t}(t) \right\|_{L^2(\mathcal{H}, \mu)}^2 dt \right) = 0. \quad (5.17)$$

Define $\phi_{n,0}^{(k)} := 0 \vee \phi_n^{(k)} = [\phi_n^{(k)}]^+$, $k, n \in \mathbb{N}$. For k fixed $\phi_{n,0}^{(k)} \in \mathcal{K}_{m,\mu}^p$ for all $m \geq n$. Standard results about weighted Sobolev spaces on \mathbb{R}^n (cf. [34], Chapter 2) guarantee that

$$\left(\frac{\partial}{\partial t} + D \right) \phi_{n,0}^{(k)} = \begin{cases} \left(\frac{\partial}{\partial t} + D \right) \phi_n^{(k)} & \text{on } \{\phi_n^{(k)} \geq 0\}, \\ 0 & \text{elsewhere.} \end{cases} \quad (5.18)$$

By using (5.15), (5.17) and (5.18) one may show that there exists a constant $C_k > 0$ such that

$$\begin{aligned} \|\phi_{n,0}^{(k)}\|_{W^{1,2}} &= \int_{[0,T] \times \mathcal{H}} \left[\left| \frac{\partial \phi_{n,0}^{(k)}}{\partial t}(t, x) \right|^2 + \|D\phi_{n,0}^{(k)}(t, x)\|_{\mathcal{H}}^2 \right] \mu(dx) dt \\ &= \int_{\{[0,T] \times \mathbb{R}^n\} \cap \{\phi_n^{(k)} \geq 0\}} \left[\left| \frac{\partial \phi_n^{(k)}}{\partial t}(t, x) \right|^2 + \|D\phi_n^{(k)}(t, x)\|_{\mathcal{H}}^2 \right] \mu_n(dx) dt \\ &\leq \int_0^T \|\phi_n^{(k)}(t)\|_{\mathcal{V}^p}^2 dt + \int_0^T \left\| \frac{\partial \phi_n^{(k)}}{\partial t}(t) \right\|_{L^2(\mathcal{H}, \mu)}^2 dt \leq C_k, \end{aligned} \quad (5.19)$$

¹The proof relies on the fact that the set of continuous functions is dense in $L^p(\mathcal{H}, \mu)$ and goes through a finite-dimensional reduction, a localization and the Stone-Weierstrass theorem.

where, as for the constants above, the dependence of C_k on k is through $\phi^{(k)}$. The convergence (5.17) and the fact that $|\phi_n^{(k)}|^+ - [\phi^{(k)}]^+| \leq |\phi_n^{(k)} - \phi^{(k)}|$ also imply

$$\lim_{n \rightarrow \infty} \int_0^T \left\| \phi_{n,0}^{(k)}(t) - [\phi^{(k)}(t)]^+ \right\|_{L^p(\mathcal{H}, \mu)}^2 dt = 0. \quad (5.20)$$

From weak relative compactness and (5.19), there exists $f \in W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu)$ and a subsequence $(\phi_{n_i(k),0}^{(k)})_{i \in \mathbb{N}}$ such that $\phi_{n_i(k),0}^{(k)} \rightharpoonup f$ in $W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu)$ as $i \rightarrow \infty$. Therefore (5.20) implies $f = [\phi^{(k)}]^+$. The notation $(n_i(k))_{i \in \mathbb{N}}$ emphasizes that such subsequence is obtained once k is fixed, hence there is one for each k .

The infinite-dimensional analogue of (5.18) (see [23], Chapter 4, Lemma 4.1) and (5.14) give

$$\begin{aligned} \|[\phi^{(k)}]^+\|_{W^{1,2}} &= \int_{[0,T] \times \mathcal{H}} \left[\left| \frac{\partial [\phi^{(k)}]^+}{\partial t}(t, x) \right|^2 + \|D[\phi^{(k)}]^+(t, x)\|_{\mathcal{H}}^2 \right] \mu(dx) dt \\ &= \int_{[0,T] \times \mathcal{H}} I_{\{\phi^{(k)} \geq 0\}}(t, x) \left[\left| \frac{\partial \phi^{(k)}}{\partial t}(t, x) \right|^2 + \|D\phi^{(k)}(t, x)\|_{\mathcal{H}}^2 \right] \mu(dx) dt \\ &\leq \int_0^T \|\phi^{(k)}(t)\|_{V^p}^2 dt + \int_0^T \left\| \frac{\partial \phi^{(k)}}{\partial t}(t) \right\|_{L^2(\mathcal{H}, \mu)}^2 dt \leq C_w, \end{aligned} \quad (5.21)$$

where $C_w > 0$ is a constant depending only on the choice of w . Again weak relative compactness and (5.21) imply that there exists $g \in W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu)$ and a subsequence $(\phi^{(k_j)})_{j \in \mathbb{N}}$ such that $[\phi^{(k_j)}]^+ \rightharpoonup g$ as $j \rightarrow \infty$ in $W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu)$. From (5.14) also $[\phi^{(k_j)}]^+ \rightarrow [w]^+$, as $j \rightarrow \infty$ in $L^2(0, T; L^p(\mathcal{H}, \mu))$ and hence $g = [w]^+ = w$.

Now consider the double indexed sequence $(\phi_{n_i(k_j),0}^{(k_j)})_{i,j \in \mathbb{N}}$, relabel the indexes to obtain $(\phi_{n,k}^{(k)})_{k,n \in \mathbb{N}}$ and set $w_{k,n} := \phi_{n,0}^{(k)}$. Taking the limits, first with respect to n and then with respect to k , the following hold

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} w_{k,n} = w, \quad \text{weakly in } W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu),$$

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} w_{k,n} = w, \quad \text{strongly in } L^2(0, T; L^p(\mathcal{H}, \mu)).$$

□

Now, consider the map $t \mapsto \|[\phi^{(k)}(t)]^+\|_{L^2(\mathcal{H}, \mu)}^2$. The next lemma provides an important regularity result and its proof is in Appendix D.

Lemma 5.3. *There exists positive constants C_1 and C_2 independent of k and such that*

$$\sup_{0 \leq t \leq T} \|[\phi^{(k)}(t)]^+\|_{L^2(\mathcal{H}, \mu)}^2 \leq C_1 \quad (5.22)$$

$$\|[\phi^{(k)}(t)]^+ - [\phi^{(k)}(s)]^+\|_{L^2(\mathcal{H}, \mu)}^2 \leq C_2 |t - s|, \quad t, s \in [0, T]. \quad (5.23)$$

Notice that (5.22) and (5.23) hold with the same constants when we replace $\phi^{(k)}$ by $\phi_n^{(k)}$. Lemma 5.3 and the Ascoli-Arzelà's theorem give the following

Corollary 5.4. *Let $w_{k,n}$ be as in Theorem 5.2. Then taking the limits, first with respect to n and then with respect to k , along a suitable subsequence, the following holds*

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \left| \|w_{k,n}(t)\|_{L^2(\mathcal{H}, \mu)}^2 - \|w(t)\|_{L^2(\mathcal{H}, \mu)}^2 \right| = 0. \quad (5.24)$$

Recall (3.2), set $u_\alpha := \mathcal{U}_\alpha - \Theta$ and denote by $(\cdot, \cdot)_\mu$ the scalar product in $L^2(\mathcal{H}, \mu)$.

Theorem 5.5. *For every $2 < p < \infty$ the function u_α is a solution of the weak variational problem on \mathcal{H}*

$$\left\{ \begin{array}{l} u \in L^2(0, T; \mathcal{V}^p); \quad u(T, x) = 0, \quad x \in \mathcal{H}; \quad u(t, x) \geq 0, \quad (t, x) \in [0, T] \times \mathcal{H}; \\ \int_0^T \left[-\left(\frac{\partial w}{\partial t}, w - u\right)_\mu + a_\mu^{(\alpha)}(u, w - u) - (f_\alpha, w - u)_\mu \right] dt + \frac{1}{2} \|w(T)\|_{L^2(\mathcal{H}, \mu)}^2 \geq 0 \end{array} \right. \quad (5.25)$$

for all $w \in \mathcal{K}_\mu^p$.

Moreover, $u_\alpha \in C([0, T] \times \mathcal{H})$.

Proof. The continuity of u_α is a consequence of Theorem 3.10 and Proposition B.1. For arbitrary $w \in \mathcal{K}_\mu^p$ take the corresponding approximating sequence $(w_{k,n})_{k,n \in \mathbb{N}}$ given by Theorem 5.2 and Corollary 5.4. For $k \in \mathbb{N}$ arbitrary but fixed, Theorems 4.3 and 5.2 and Remark 2.8 guarantee

$$\begin{aligned} \int_0^T \left[-\left(\frac{\partial w_{k,n}}{\partial t}, w_{k,n} - u_\alpha^{(m)}\right)_\mu + a_\mu^{(\alpha, m)}(u_\alpha^{(m)}, w_{k,n} - u_\alpha^{(m)}) - (f_{\alpha, m}, w_{k,n} - u_\alpha^{(m)})_\mu \right] dt \\ + \frac{1}{2} \|w_{k,n}(T)\|_{L^2(\mathcal{H}, \mu)}^2 \geq 0, \end{aligned}$$

for $m \geq n$. In the limit as $m \rightarrow \infty$, Proposition 5.1, equations (5.8) and (5.10) and arguments similar to those used in the proof of Theorem 4.3 give

$$\int_0^T \left[-\left(\frac{\partial w_{k,n}}{\partial t}, w_{k,n} - u_\alpha\right)_\mu + a_\mu^{(\alpha)}(u_\alpha, w_{k,n} - u_\alpha) - (f_\alpha, w_{k,n} - u_\alpha)_\mu \right] dt + \frac{1}{2} \|w_{k,n}(T)\|_{L^2(\mathcal{H}, \mu)}^2 \geq 0.$$

The proof now follows from Theorem 5.2 and Corollary 5.4. \square

5.2 The optimal stopping time

In this section the existence of an optimal stopping time for \mathcal{U}_α of (3.2) is obtained by purely probabilistic considerations. Two preliminary lemmas provide the basic tools needed to prove Theorem 5.8 below. Given $(t, x) \in [0, T] \times \mathcal{H}$, let $\tau_{\alpha, n}^*(t, x)$ be as in (4.20) and let $\tau_\alpha^*(t, x)$ be

$$\tau_\alpha^*(t, x) := \inf\{s \geq t : \mathcal{U}_\alpha(s, X_s^{(\alpha)t, x}) = \Theta(s, X_s^{(\alpha)t, x})\} \wedge T. \quad (5.26)$$

Lemma 5.6. *There exists a subsequence $(\tau_{\alpha, n_j}^*(t, x))_{j \in \mathbb{N}}$ such that*

$$\lim_{j \rightarrow \infty} (\tau_\alpha^*(t, x) \wedge \tau_{\alpha, n_j}^*(t, x))(\omega) = \tau_\alpha^*(t, x)(\omega), \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (5.27)$$

Proof. There is no loss of generality if we consider the diffusions $X^{(\alpha)x}$ and $X^{(\alpha)x; n}$ starting at time zero as all results remain true for arbitrary initial time t . The proof of this Lemma is adapted from [3], Chapter 3, Section 3, Theorem 3.7 (cf. in particular p. 322).

Using Proposition 3.5, fix $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and a subsequence $\mathcal{N} := (n_j)_{j \in \mathbb{N}}$ such that

$$\lim_{j \rightarrow \infty} \sup_{0 \leq t \leq T} \|X_t^{(\alpha)x; n_j}(\omega) - X_t^{(\alpha)x}(\omega)\|_{\mathcal{H}} \rightarrow 0, \quad \forall \omega \in \Omega_0. \quad (5.28)$$

Since the starting point $x_0 \in \mathcal{H}$ is fixed, to simplify the notation in the rest of the proof, we shall write $\tau_{\alpha, n}^*$ and τ_α^* instead of $\tau_{\alpha, n}^*(0, x_0)$ and $\tau_\alpha^*(0, x_0)$, respectively. The limit (5.27) is trivial if

$\omega' \in \Omega_0$ is such that $\tau_\alpha^*(\omega') = 0$. On the other hand, if $\omega' \in \Omega_0$ is such that $\tau_\alpha^*(\omega') > \delta$ for some $\delta = \delta_{x_0} > 0$, then by (5.26)

$$\mathcal{U}_\alpha(t, X_t^{(\alpha)x_0}(\omega')) > \Theta(t, X_t^{(\alpha)x_0}(\omega')), \quad t \in [0, \tau_\alpha^*(\omega') - \delta].$$

Since the map $t \mapsto X_t^{(\alpha)x_0}(\omega')$ is continuous and $[0, \tau_\alpha^*(\omega') - \delta]$ is a compact set it follows that the set $\chi^\delta(\omega') := \{y \in \mathcal{H} : y = X_t^{(\alpha)x_0}(\omega'), t \in [0, \tau_\alpha^*(\omega') - \delta]\}$ is a compact subset of \mathcal{H} . Therefore the continuous map $(t, x) \mapsto \mathcal{U}_\alpha(t, x) - \Theta(t, x)$ (cf. Theorem 3.10) attains its minimum on $[0, \tau_\alpha^*(\omega') - \delta] \times \chi^\delta(\omega')$, call it $\rho(\delta, \omega') > 0$. Then

$$\mathcal{U}_\alpha(t, X_t^{(\alpha)x_0}(\omega')) \geq \Theta(t, X_t^{(\alpha)x_0}(\omega')) + \rho(\delta, \omega'), \quad t \in [0, \tau_\alpha^*(\omega') - \delta]. \quad (5.29)$$

Recall from Theorem 3.9 and (3.11) that $\mathcal{U}_\alpha^{(n)}$ and $\Theta^{(n)}$ converge respectively to \mathcal{U}_α and Θ , uniformly on compact subsets of $[0, T] \times \mathcal{H}$. Therefore there exists $n_\rho = n(\rho(\delta, \omega')) \in \mathcal{N}$, $n_\rho > 0$ large enough such that

$$\mathcal{U}_\alpha^{(n_\rho)}(t, x^{(n_\rho)}) \geq \mathcal{U}_\alpha(t, x) - \frac{1}{4}\rho(\delta, \omega'), \quad (t, x) \in [0, \tau_\alpha^*(\omega') - \delta] \times \chi^\delta(\omega'), \quad (5.30)$$

$$\Theta^{(n_\rho)}(t, x^{(n_\rho)}) \leq \Theta(t, x) + \frac{1}{4}\rho(\delta, \omega'), \quad (t, x) \in [0, \tau_\alpha^*(\omega') - \delta] \times \chi^\delta(\omega'), \quad (5.31)$$

and

$$\sup_{0 \leq t \leq T} \|X^{(\alpha)x_0; n_\rho}(\omega') - X^{(\alpha)x_0}(\omega')\|_{\mathcal{H}} \leq \frac{1}{2L_{\mathcal{U}} \vee L_{\Theta}} \rho(\delta, \omega'). \quad (5.32)$$

Now (5.29), (5.30) and (5.31) imply

$$\mathcal{U}_\alpha^{(n_\rho)}(t, P_{n_\rho} X_t^{(\alpha)x_0}(\omega')) \geq \Theta^{(n_\rho)}(t, P_{n_\rho} X_t^{(\alpha)x_0}(\omega')) + \frac{1}{2}\rho(\delta, \omega'), \quad t \in [0, \tau_\alpha^*(\omega') - \delta]. \quad (5.33)$$

On the other hand Proposition 2.7 and the fact that $P_{n_\rho} X^{(\alpha)x_0; n_\rho} = X^{(\alpha)x_0; n_\rho}$ imply

$$\sup_{0 \leq t \leq T} \left| \mathcal{U}_\alpha^{(n_\rho)}(t, P_{n_\rho} X_t^{(\alpha)x_0}(\omega')) - \mathcal{U}_\alpha^{(n_\rho)}(t, X_t^{(\alpha)x_0; n_\rho}(\omega')) \right| \leq L_{\mathcal{U}} \sup_{0 \leq t \leq T} \|X_t^{(\alpha)x_0; n_\rho}(\omega') - X_t^{(\alpha)x_0}(\omega')\|_{\mathcal{H}} \quad (5.34)$$

and

$$\sup_{0 \leq t \leq T} \left| \Theta^{(n_\rho)}(t, P_{n_\rho} X_t^{(\alpha)x_0}(\omega')) - \Theta^{(n_\rho)}(t, X_t^{(\alpha)x_0; n_\rho}(\omega')) \right| \leq L_{\Theta} \sup_{0 \leq t \leq T} \|X_t^{(\alpha)x_0; n_\rho}(\omega') - X_t^{(\alpha)x_0}(\omega')\|_{\mathcal{H}} \quad (5.35)$$

which, together with (5.32) and (5.33), imply

$$\mathcal{U}_\alpha^{(n_\rho)}(t, X_t^{(\alpha)x_0; n_\rho}(\omega')) > \Theta^{(n_\rho)}(t, X_t^{(\alpha)x_0; n_\rho}(\omega')), \quad t \in [0, \tau_\alpha^*(\omega') - \delta].$$

It follows that $\tau_{\alpha, n_\rho}^*(\omega') > \tau_\alpha^*(\omega') - \delta$. Notice that $\rho(\delta, \omega') \rightarrow 0$ as $\delta \rightarrow 0$ and hence $n_\rho \rightarrow \infty$. Therefore $\tau_{\alpha, n_\rho}^*(\omega') \wedge \tau_\alpha^*(\omega') \rightarrow \tau_\alpha^*(\omega')$ as $n_\rho \rightarrow \infty$, which is equivalent to say (5.27) holds along a subsequence. \square

Lemma 5.7. *For $s \in [0, T]$ let Y_1 and Y_2 be two \mathcal{F}_s -measurable random variables taking values on \mathcal{H} . Then*

$$|\mathcal{U}_\alpha^{(n)}(s, Y_1^{(n)}) - \mathcal{U}_\alpha(s, Y_2)| \leq L_{\Theta} \mathbb{E} \left\{ \sup_{s \leq u \leq T} \|X_u^{(\alpha)s, Y_1; n} - X_u^{(\alpha)s, Y_2}\|_{\mathcal{H}} \middle| \mathcal{F}_s \right\}, \quad \mathbb{P}\text{-a.s.}$$

Proof. From Assumption 2.4 and equations (3.2) and (3.12) follows that

$$\begin{aligned} |\mathcal{U}_\alpha^{(n)}(s, Y_1^{(n)}) - \mathcal{U}_\alpha(s, Y_2)| &\leq \operatorname{ess\,sup}_{s \leq \tau \leq T} \mathbb{E} \left\{ \left| \Theta^{(n)}(\tau, X_\tau^{(\alpha)s, Y_1; n}) - \Theta(\tau, X_\tau^{(\alpha)s, Y_2}) \right| \middle| \mathcal{F}_s \right\} \\ &\leq L_\Theta \mathbb{E} \left\{ \sup_{s \leq u \leq T} \|X_u^{(\alpha)s, Y_1; n} - X_u^{(\alpha)s, Y_2}\|_{\mathcal{H}} \middle| \mathcal{F}_s \right\}. \end{aligned}$$

□

Theorem 5.8. *The optimal stopping time of (3.2) is $\tau_\alpha^*(t, x)$ as defined in (5.26).*

Proof. Given the initial data (t, x) we adopt the simplified notation notation used in the proof of Lemma 5.6; that is we set $\tau_\alpha^* := \tau_\alpha^*(t, x)$ and $\tau_{\alpha, n}^* := \tau_{\alpha, n}^*(t, x)$. By Remark 4.4 we have

$$\mathcal{U}_\alpha^{(n)}(t, x^{(n)}) = \mathbb{E} \left\{ \mathcal{U}_\alpha^{(n)}(\tau_\alpha^* \wedge \tau_{\alpha, n}^*, X_{\tau_\alpha^* \wedge \tau_{\alpha, n}^*}^{(\alpha)t, x; n}) \right\}. \quad (5.36)$$

In (5.36), take the subsequence $(n_j)_{j \in \mathbb{N}}$ of Lemma 5.6. Then Theorem 3.9 guarantees the convergence of $\mathcal{U}_\alpha^{(n_j)}(t, x^{(n_j)})$ to $\mathcal{U}_\alpha(t, x)$ as $j \rightarrow \infty$. On the other hand

$$\begin{aligned} &\left| \mathbb{E} \left\{ \mathcal{U}_\alpha^{(n_j)}(\tau_\alpha^* \wedge \tau_{\alpha, n_j}^*, X_{\tau_\alpha^* \wedge \tau_{\alpha, n_j}^*}^{(\alpha)t, x; n_j}) - \mathcal{U}_\alpha(\tau_\alpha^*, X_{\tau_\alpha^*}^{(\alpha)t, x}) \right\} \right| \\ &\leq \mathbb{E} \left\{ \sup_{t \leq s \leq T} \left| \mathcal{U}_\alpha^{(n_j)}(s, X_s^{(\alpha)t, x; n_j}) - \mathcal{U}_\alpha^{(n_j)}(s, P_{n_j} X_s^{(\alpha)t, x}) \right| \right\} \quad (5.37) \\ &\quad + \mathbb{E} \left\{ \sup_{t \leq s \leq T} \left| \mathcal{U}_\alpha^{(n_j)}(s, P_{n_j} X_s^{(\alpha)t, x}) - \mathcal{U}_\alpha(s, X_s^{(\alpha)t, x}) \right| \right\} \\ &\quad + \mathbb{E} \left\{ \left| \mathcal{U}_\alpha(\tau_\alpha^* \wedge \tau_{\alpha, n_j}^*, X_{\tau_\alpha^* \wedge \tau_{\alpha, n_j}^*}^{(\alpha)t, x}) - \mathcal{U}_\alpha(\tau_\alpha^*, X_{\tau_\alpha^*}^{(\alpha)t, x}) \right| \right\}, \end{aligned}$$

and below we will estimate the three terms on the right hand side of (5.37).

Recall that $X^{(\alpha)t, x; n_j} = P_{n_j} X^{(\alpha)t, x; n_j}$ and use Lemma 5.7, the Markov property and Jensen's inequality and to estimate the first term. In fact there exists a constant $C > 0$ such that

$$\begin{aligned} &\mathbb{E} \left\{ \sup_{t \leq s \leq T} \left| \mathcal{U}_\alpha^{(n_j)}(s, X_s^{(\alpha)t, x; n_j}) - \mathcal{U}_\alpha^{(n_j)}(s, P_{n_j} X_s^{(\alpha)t, x}) \right| \right\} \\ &\leq C \left(\mathbb{E} \left\{ \sup_{t \leq s \leq T} \mathbb{E} \left\{ \sup_{t \leq u \leq T} \|X_u^{(\alpha)t, x; n_j} - X_u^{(\alpha)t, x}\|_{\mathcal{H}}^2 \middle| \mathcal{F}_s \right\} \right\} \right)^{\frac{1}{2}}, \quad (5.38) \end{aligned}$$

and that goes to zero as $j \rightarrow \infty$ by Doob's inequality and Proposition 3.5.

For the second term on the right hand side of (5.37) apply again Lemma 5.7 to obtain

$$\begin{aligned} &\mathbb{E} \left\{ \sup_{t \leq s \leq T} \left| \mathcal{U}_\alpha^{(n_j)}(s, P_{n_j} X_s^{(\alpha)t, x}) - \mathcal{U}_\alpha(s, X_s^{(\alpha)t, x}) \right| \right\} \\ &\leq L_\Theta \mathbb{E} \left\{ \sup_{t \leq s \leq T} \mathbb{E} \left\{ \sup_{s \leq u \leq T} \|X_u^{(\alpha)s, P_{n_j} X_s^{(\alpha)t, x; n_j}} - X_u^{(\alpha)s, X_s^{(\alpha)t, x}}\|_{\mathcal{H}} \middle| \mathcal{F}_s \right\} \right\}, \end{aligned}$$

and hence

$$\begin{aligned} &\mathbb{E} \left\{ \sup_{t \leq s \leq T} \left| \mathcal{U}_\alpha^{(n_j)}(s, P_{n_j} X_s^{(\alpha)t, x}) - \mathcal{U}_\alpha(s, X_s^{(\alpha)t, x}) \right| \right\} \quad (5.39) \\ &\leq L_\Theta \left(\mathbb{E} \left\{ \sup_{t \leq s \leq T} \mathbb{E} \left\{ \sup_{s \leq u \leq T} \|X_u^{(\alpha)s, P_{n_j} X_s^{(\alpha)t, x; n_j}} - X_u^{(\alpha)s, X_s^{(\alpha)t, x}}\|_{\mathcal{H}}^2 \middle| \mathcal{F}_s \right\} \right\} \right)^{\frac{1}{2}} \end{aligned}$$

by using Jensen's inequality twice gives. Notice that $X^{(\alpha)s, P_{n_j} X_s^{(\alpha)t, x}; n_j} = X^{(\alpha)s, X_s^{(\alpha)t, x}; n_j}$ by (3.5). Usual arguments (see for example (3.8) in the proof of Proposition 3.5) and the Markov property give

$$\begin{aligned}
& \mathbb{E} \left\{ \sup_{s \leq u \leq T} \|X_u^{(\alpha)s, P_{n_j} X_s^{(\alpha)t, x}; n_j} - X_u^{(\alpha)s, X_s^{(\alpha)t, x}}\|_{\mathcal{H}}^2 \middle| \mathcal{F}_s \right\} \\
& \leq \bar{C} \left(\|P_{n_j} X_s^{(\alpha)t, x} - X_s^{(\alpha)t, x}\|_{\mathcal{H}}^2 + \int_s^T \mathbb{E} \left\{ \|(I - P_{n_j}) A_{\alpha} X_u^{(\alpha)s, X_s^{(\alpha)t, x}}\|_{\mathcal{H}}^2 \middle| \mathcal{F}_s \right\} du \right. \\
& \quad \left. + \int_s^T \mathbb{E} \left\{ \|(I - P_{n_j}) \sigma(X_u^{(\alpha)s, X_s^{(\alpha)t, x}})\|_{\mathcal{H}}^2 \middle| \mathcal{F}_s \right\} du + n_j^2 \epsilon_{n_j}^2 T \right) \\
& \leq \bar{C} \left(\|P_{n_j} X_s^{(\alpha)t, x} - X_s^{(\alpha)t, x}\|_{\mathcal{H}}^2 + \mathbb{E} \left\{ \int_t^T \|(I - P_{n_j}) A_{\alpha} X_u^{(\alpha)t, x}\|_{\mathcal{H}}^2 du \middle| \mathcal{F}_s \right\} \right. \\
& \quad \left. + \mathbb{E} \left\{ \int_t^T \|(I - P_{n_j}) \sigma(X_u^{(\alpha)t, x})\|_{\mathcal{H}}^2 du \middle| \mathcal{F}_s \right\} + n_j^2 \epsilon_{n_j}^2 T \right),
\end{aligned} \tag{5.40}$$

for a suitable constant $\bar{C} > 0$. Hence (5.39) gives

$$\begin{aligned}
& \mathbb{E} \left\{ \sup_{t \leq s \leq T} \left| \mathcal{U}_{\alpha}^{(n_j)}(s, P_{n_j} X_s^{(\alpha)t, x}) - \mathcal{U}_{\alpha}(s, X_s^{(\alpha)t, x}) \right| \right\} \\
& \leq L_{\Theta} \bar{C}^{\frac{1}{2}} \left(\mathbb{E} \left\{ \sup_{t \leq s \leq T} \|P_{n_j} X_s^{(\alpha)t, x} - X_s^{(\alpha)t, x}\|_{\mathcal{H}}^2 \right\} \right. \\
& \quad + \mathbb{E} \left\{ \sup_{t \leq s \leq T} \mathbb{E} \left\{ \int_t^T \|(I - P_{n_j}) A_{\alpha} X_u^{(\alpha)t, x}\|_{\mathcal{H}}^2 du \middle| \mathcal{F}_s \right\} \right\} \\
& \quad \left. + \mathbb{E} \left\{ \sup_{t \leq s \leq T} \mathbb{E} \left\{ \int_t^T \|(I - P_{n_j}) \sigma(X_u^{(\alpha)t, x})\|_{\mathcal{H}}^2 du \middle| \mathcal{F}_s \right\} \right\} + n_j^2 \epsilon_{n_j}^2 T \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.41}$$

The first term on the right hand side of (5.41) converges to zero as $j \rightarrow \infty$ by dominated convergence and Dini's Theorem. The last term goes to zero by (3.6). To show that the remaining terms go to zero as well, define the non-negative, square integrable martingales

$$\begin{aligned}
M_s^j &:= \mathbb{E} \left\{ \int_t^T \|(I - P_{n_j}) A_{\alpha} X_u^{(\alpha)t, x}\|_{\mathcal{H}}^2 du \middle| \mathcal{F}_s \right\}, \\
N_s^j &:= \mathbb{E} \left\{ \int_t^T \|(I - P_{n_j}) \sigma(X_u^{(\alpha)t, x})\|_{\mathcal{H}}^2 du \middle| \mathcal{F}_s \right\},
\end{aligned}$$

and apply Jensen's inequality and Doob's inequality to obtain

$$\mathbb{E} \left\{ \sup_{t \leq s \leq T} |M_s^j| \right\} \leq 2 \left(\mathbb{E} \left\{ \int_t^T \|(I - P_{n_j}) A_{\alpha} X_u^{(\alpha)t, x}\|_{\mathcal{H}}^4 du \right\} \right)^{\frac{1}{2}}, \tag{5.42}$$

$$\mathbb{E} \left\{ \sup_{t \leq s \leq T} |N_s^j| \right\} \leq 2 \left(\mathbb{E} \left\{ \int_t^T \|(I - P_{n_j}) \sigma(X_u^{(\alpha)t, x})\|_{\mathcal{H}}^4 du \right\} \right)^{\frac{1}{2}}. \tag{5.43}$$

The desired results follow by dominated convergence.

As for the last term in (5.37), it goes to zero as $j \rightarrow \infty$ by dominated convergence, Lemma 5.6 and the continuity of the \mathcal{U}_{α} .

In conclusion by taking the limits along the subsequence $(n_j)_{j \in \mathbb{N}}$, in (5.36) we obtain

$$\mathcal{U}_{\alpha}(t, x) = \mathbb{E} \left\{ \mathcal{U}_{\alpha}(\tau_{\alpha}^*, X_{\tau_{\alpha}^*}^{(\alpha)t, x}) \right\} = \mathbb{E} \left\{ \Theta(\tau_{\alpha}^*, X_{\tau_{\alpha}^*}^{(\alpha)t, x}) \right\}, \tag{5.44}$$

which proves the optimality of τ_{α}^* . \square

5.3 Removal of the Yosida approximation

The function u_α in Theorem 5.5 solves the variational inequality associated to the Yosida approximation A_α of the unbounded operator A . In this section we study the limiting behavior, as $\alpha \rightarrow \infty$, of u_α and of the corresponding variational inequality by adopting both probabilistic and analytical tools. The family $(u_\alpha)_{\alpha>0}$ is uniformly bounded in $L^2(0, T; W^{1,2}(\mathcal{H}, \mu))$ (cf. Proposition B.5). By Theorem 3.2 it is not hard to prove the following result.

Proposition 5.9. *The sequence u_α converges to \hat{u} as $\alpha \rightarrow \infty$, weakly in $L^2(0, T; W^{1,2}(\mathcal{H}, \mu))$ and strongly in $L^2(0, T; L^p(\mathcal{H}, \mu))$, $1 \leq p < \infty$. Moreover \hat{u} is identified as $\hat{u} = \mathcal{U} - \Theta$.*

When $\alpha \rightarrow \infty$ the term involving A_α in the bilinear form $a_\mu^{(\alpha)}(\cdot, \cdot)$ of (5.25) converges to a suitable operator that needs to be fully characterized. Let $w \in L^2(0, T; \mathcal{V}^p)$ be given and define the linear functional $T_A^{(n)}(w, \cdot) \in [L^2(0, T; \mathcal{V}^p)]^*$ by

$$T_A^{(n)}(w, u) := \int_0^T \int_{\mathcal{H}} \langle A P_n x, Du \rangle_{\mathcal{H}} w \mu(dx) dt, \quad u \in L^2(0, T; \mathcal{V}^p). \quad (5.45)$$

It is easy to show that $T_A^{(n)}(w, \cdot)$ is continuous by (5.3) and that $(T_A^{(n)}(w, \cdot))_{n \in \mathbb{N}}$ is a Cauchy sequence in $[L^2(0, T; \mathcal{V}^p)]^*$. In fact for $n > m$ arguments similar to those in the proof of Theorem 4.2 give

$$\begin{aligned} |T_A^{(n)}(w, u) - T_A^{(m)}(w, u)| &= \left| \sum_{j=m+1}^n \int_0^T \int_{\mathcal{H}} \langle A \varphi_j, Du \rangle_{\mathcal{H}} x_j w \mu(dx) dt \right| \\ &\leq C_p \left(\int_0^T \|u(t)\|_p^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \|w(t)\|_p^2 dt \right)^{\frac{1}{2}} \sum_{j=m+1}^n \|A \varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j}, \end{aligned}$$

and hence

$$\|T_A^{(n)}(w, \cdot) - T_A^{(m)}(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} \leq C_p \left(\int_0^T \|w(t)\|_p^2 dt \right)^{\frac{1}{2}} \sum_{j=m+1}^n \|A \varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j}. \quad (5.46)$$

Since $\sum_{j=1}^n \|A \varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j}$ converges as $n \rightarrow \infty$ (cf. Assumption 2.3), (5.46) goes to zero as $m, n \rightarrow \infty$ and $(T_A^{(n)}(w, \cdot))_{n \in \mathbb{N}}$ is Cauchy in $[L^2(0, T; \mathcal{V}^p)]^*$. Therefore, by completeness of $[L^2(0, T; \mathcal{V}^p)]^*$ there exists $\hat{T}_A(w, \cdot) \in [L^2(0, T; \mathcal{V}^p)]^*$ such that $T_A^{(n)}(w, \cdot) \rightarrow \hat{T}_A(w, \cdot)$ as $n \rightarrow \infty$.

Recall that the set $\mathcal{E}_A([0, T] \times \mathcal{H})$ of (5.13) is dense in $L^2(0, T; \mathcal{V}^p)$. Then it suffices to characterize the explicit form of $\hat{T}_A(w, \cdot)$ on that subset. Notice also that $A^* Du \in L^2(0, T; L^2(\mathcal{H}, \mu))$ for $u \in \mathcal{E}_A([0, T] \times \mathcal{H})$ and consequently

$$\int_0^T \int_{\mathcal{H}} \langle A P_n x, Du \rangle_{\mathcal{H}} w \mu(dx) dt = \int_0^T \int_{\mathcal{H}} \langle P_n x, A^* Du \rangle_{\mathcal{H}} w \mu(dx) dt.$$

Now dominated convergence gives

$$\lim_{n \rightarrow \infty} T_A^{(n)}(w, u) = \int_0^T \int_{\mathcal{H}} \langle x, A^* Du \rangle_{\mathcal{H}} w \mu(dx) dt, \quad \text{for } u \in \mathcal{E}_A([0, T] \times \mathcal{H}). \quad (5.47)$$

So (5.47) defines a linear functional $T_A(w, \cdot)$ whose domain $D(T_A(w, \cdot))$ contains $\mathcal{E}_A([0, T] \times \mathcal{H})$ and is dense in $L^2(0, T; \mathcal{V}^p)$. Since the estimate (5.3) is uniform with respect to $n \in \mathbb{N}$ we obtain

$$|T_A(w, u)| \leq C_{\mu, \gamma, p} \|w\|_{L^2(0, T; \mathcal{V}^p)} \|u\|_{L^2(0, T; \mathcal{V}^p)}, \quad \text{for } u \in \mathcal{E}_A([0, T] \times \mathcal{H}). \quad (5.48)$$

By Hahn-Banach theorem $T_A(w, \cdot)$ is extended to the whole space $L^2(0, T; \mathcal{V}^p)$ and the extended functional is denoted by $\bar{T}_A(w, \cdot)$. From density arguments and (5.3) one may check that

$$\hat{T}_A(w, \cdot) := \lim_{n \rightarrow \infty} T_A^{(n)}(w, \cdot) = \bar{T}_A(w, \cdot) \quad \text{in } [L^2(0, T; \mathcal{V}^p)]^*. \quad (5.49)$$

We are now able to identify the limit of the Yosida approximation. Define $T_A^{(\alpha)}(w, \cdot) \in [L^2(0, T; \mathcal{V}^p)]^*$ by

$$T_A^{(\alpha)}(w, u) := \int_0^T \int_{\mathcal{H}} \langle A_\alpha x, Du \rangle_{\mathcal{H}} w \mu(dx) dt, \quad \text{for } u \in L^2(0, T; \mathcal{V}^p), \quad (5.50)$$

and, as in (5.45), set

$$T_A^{(\alpha, n)}(w, u) := \int_0^T \int_{\mathcal{H}} \langle A_\alpha P_n x, Du \rangle_{\mathcal{H}} w \mu(dx) dt. \quad (5.51)$$

The same arguments leading to (5.46) give

$$\|T_A^{(\alpha, n)}(w, \cdot) - T_A^{(\alpha)}(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} \leq C_p \sum_{j=n+1}^{\infty} \|A\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j} \left(\int_0^T \|w(t)\|_p^2 dt \right)^{\frac{1}{2}}, \quad (5.52)$$

then the uniform convergence follows,

$$\lim_{n \rightarrow \infty} \sup_{\alpha > 0} \|T_A^{(\alpha, n)}(w, \cdot) - T_A^{(\alpha)}(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} = 0. \quad (5.53)$$

Proposition 5.10. *With $\bar{T}_A(w, \cdot)$ given by (5.49), it holds that*

$$\lim_{\alpha \rightarrow \infty} \|T_A^{(\alpha)}(w, \cdot) - \bar{T}_A(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} = 0. \quad (5.54)$$

Proof. Recall (5.45) and (5.51). Fix $\varepsilon > 0$, then by (5.49) and (5.53) there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned} \|T_A^{(\alpha)}(w, \cdot) - \bar{T}_A(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} &\leq \|T_A^{(\alpha)}(w, \cdot) - T_A^{(\alpha, n_\varepsilon)}(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} \\ &\quad + \|T_A^{(\alpha, n_\varepsilon)}(w, \cdot) - T_A^{(n_\varepsilon)}(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} \\ &\quad + \|T_A^{(n_\varepsilon)}(w, \cdot) - \bar{T}_A(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} \\ &< \varepsilon + \|T_A^{(\alpha, n_\varepsilon)}(w, \cdot) - T_A^{(n_\varepsilon)}(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*}. \end{aligned} \quad (5.55)$$

Moreover,

$$\|T_A^{(\alpha, n_\varepsilon)}(w, \cdot) - T_A^{(n_\varepsilon)}(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} \leq C_p \left(\int_0^T \|w(t)\|_p^2 dt \right)^{\frac{1}{2}} \sum_{j=1}^{\infty} \|(A_\alpha - A)\varphi_j\|_{\mathcal{H}} \sqrt{\lambda_j}, \quad (5.56)$$

as in (5.46). By Assumption 2.3 and the fact that $\|A_\alpha \varphi_j\|_{\mathcal{H}} \leq M \|A \varphi_j\|_{\mathcal{H}}$, $j \in \mathbb{N}$ for $M > 0$, the sum in the right-hand side of (5.56) converges uniformly with respect to $\alpha > 0$. It is well known (cf. [28]) that

$$\lim_{\alpha \rightarrow \infty} A_\alpha \varphi_j = A \varphi_j, \quad j \in \mathbb{N}, \quad (5.57)$$

and hence (5.55) together with (5.56) gives

$$\lim_{\alpha \rightarrow \infty} \|T_A^{(\alpha)}(w, \cdot) - \bar{T}_A(w, \cdot)\|_{[L^2(0, T; \mathcal{V}^p)]^*} < \varepsilon. \quad (5.58)$$

Since ε is arbitrary the proof follows. \square

Recall that for every $w \in L^2(0, T; W^{1,2}(\mathcal{H}, \mu))$ we defined the family $(\mathcal{T}_{\alpha, w})_{\alpha > 0}$ in (5.6). Hence if we set

$$\mathcal{T}_w(u) := \int_0^T a_\mu(u, w) dt, \quad u \in L^2(0, T; \mathcal{V}^p) \quad (5.59)$$

and

$$\begin{aligned} \int_0^T a_\mu(u, w) dt &:= \frac{1}{2} \int_0^T \int_{\mathcal{H}} \langle \sigma \sigma^* Du, Dw \rangle_{\mathcal{H}} \mu(dx) dt + \frac{1}{2} \int_0^T \int_{\mathcal{H}} \text{Tr}[D\sigma]_{\mathcal{H}} \langle \sigma, Du \rangle_{\mathcal{H}} w \mu(dx) dt \\ &\quad + \frac{1}{2} \int_0^T \int_{\mathcal{H}} \langle D\sigma \cdot \sigma - \sigma \sigma^* Q^{-1} x, Du \rangle_{\mathcal{H}} w \mu(dx) dt - \bar{T}_A(w, u), \end{aligned} \quad (5.60)$$

by Proposition 5.10 we obtain

$$\lim_{\alpha \rightarrow \infty} \|\mathcal{T}_{\alpha, w} - \mathcal{T}_w\|_{[L^2(0, T; \mathcal{V}^p)]^*} = 0. \quad (5.61)$$

Remark 5.11. Notice that for our gain function Θ we have $T_A^{(\alpha)}(\cdot, \Theta) \in [L^2(0, T; \mathcal{V}^p)]^*$. Arguments similar to those used in the proof of Proposition 5.10 give $T_A^{(\alpha)}(\cdot, \Theta) \rightarrow \bar{T}_A(\cdot, \Theta)$ in $[L^2(0, T; \mathcal{V}^p)]^*$ as $\alpha \rightarrow \infty$.

Define $F \in [L^2(0, T; \mathcal{V}^p)]^*$ by

$$F(w) := \int_0^T \left(\frac{\partial \Theta}{\partial t}(t) + \frac{1}{2} \text{Tr}[\sigma \sigma^* D^2 \Theta], w \right)_{\mu} dt + \bar{T}_A(w, \Theta), \quad \forall w \in L^2(0, T; \mathcal{V}^p). \quad (5.62)$$

Then, with f_α as in (5.9), from dominated convergence, Assumption 2.4, (5.4), and Proposition 5.10, follows that

$$\lim_{\alpha \rightarrow \infty} \left\| \int_0^T (f_\alpha, \cdot)_{\mu} dt - F(\cdot) \right\|_{[L^2(0, T; \mathcal{V}^p)]^*} = 0. \quad (5.63)$$

We now obtain the extension of Theorem 5.5 to the case of the unbounded operator A .

Theorem 5.12. For every $2 < p < \infty$ the function \hat{u} of Proposition 5.9 is a solution of the weak variational problem

$$\left\{ \begin{array}{l} u \in L^2(0, T; \mathcal{V}^p); \quad u(T, x) = 0, x \in \mathcal{H}; \quad u(t, x) \geq 0, (t, x) \in [0, T] \times \mathcal{H}; \\ \int_0^T \left[- \left(\frac{\partial w}{\partial t}, w - u \right)_{\mu} + a_\mu(u, w - u) \right] dt - F(w - u) + \frac{1}{2} \|w(T)\|_{L^2(\mathcal{H}, \mu)}^2 \geq 0 \\ \text{for all } w \in \mathcal{K}_\mu^p. \end{array} \right. \quad (5.64)$$

Moreover, $\hat{u} \in C([0, T] \times \mathcal{H})$.

Proof. For arbitrary $w \in \mathcal{K}_\mu^p$ take the limit as $\alpha \rightarrow \infty$ in (5.25) by exploiting Proposition 5.9, (5.61), (5.63) and the fact that

$$\int_0^T a_\mu^{(\alpha)}(u_\alpha - \hat{u}, u_\alpha - \hat{u}) dt \geq -\Lambda_p \int_0^T \|u_\alpha - \hat{u}\|_{L^p(\mathcal{H}, \mu)} dt, \quad (5.65)$$

which is the analogue of (4.31). The variational inequality now follows by arguments even simpler than those adopted in the proof of Theorem 5.5 since here w is fixed. Continuity of the solution is a consequence of Theorem 3.3. \square

The optimal stopping time of \mathcal{U} is found by probabilistic arguments as in Section 5.2. For $(t, x) \in [0, T] \times \mathcal{H}$, let $\tau_\alpha^*(t, x)$ be defined as in (5.26) and define $\tau^*(t, x)$ by

$$\tau^*(t, x) := \inf\{s \geq t : \mathcal{U}(s, X_s^{t,x}) = \Theta(s, X_s^{t,x})\} \wedge T. \quad (5.66)$$

Lemma 5.13. *There exists a subsequence $(\tau_{\alpha_j}^*(t, x))_{j \in \mathbb{N}}$ such that the following convergence holds*

$$\lim_{j \rightarrow \infty} (\tau^*(t, x) \wedge \tau_{\alpha_j}^*(t, x))(\omega) = \tau^*(t, x)(\omega), \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \quad (5.67)$$

Proof. The proof is as in Lemma 5.6 and follows from Theorem 3.3 and Proposition 3.1. \square

Theorem 5.14. *The stopping time $\tau^*(t, x)$ in (5.66) is optimal for $\mathcal{U}(t, x)$ of.*

Proof. Simplify the notation by setting $\tau^* = \tau^*(t, x)$. Substitute $\tau^* \wedge \tau_\alpha^*$ to τ_α^* in (5.36) and proceed as in the proof of Theorem 5.8 to obtain

$$\mathcal{U}_\alpha(t, x) = \mathbb{E} \left\{ \mathcal{U}_\alpha(\tau^* \wedge \tau_\alpha^*, X_{\tau^* \wedge \tau_\alpha^*}^{(\alpha)t,x}) \right\}. \quad (5.68)$$

Consider the subsequence $(\mathcal{U}_{\alpha_j})_{j \in \mathbb{N}}$ corresponding to the subsequence of stopping times in Lemma 5.13 and along that take the limit in (5.68). For the term on the right hand side we have

$$\begin{aligned} & \left| \mathbb{E} \left\{ \mathcal{U}_{\alpha_j}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{(\alpha_j)t,x}) - \mathcal{U}(\tau^*, X_{\tau^*}^{t,x}) \right\} \right| \\ & \leq \mathbb{E} \left\{ \left| \mathcal{U}_{\alpha_j}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{(\alpha_j)t,x}) - \mathcal{U}_{\alpha_j}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{t,x}) \right| \right\} \\ & \quad + \mathbb{E} \left\{ \left| \mathcal{U}_{\alpha_j}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{t,x}) - \mathcal{U}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{t,x}) \right| \right\} \\ & \quad + \mathbb{E} \left\{ \left| \mathcal{U}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{t,x}) - \mathcal{U}(\tau^*, X_{\tau^*}^{t,x}) \right| \right\}. \end{aligned} \quad (5.69)$$

For the first term we obtain

$$\mathbb{E} \left\{ \left| \mathcal{U}_{\alpha_j}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{(\alpha_j)t,x}) - \mathcal{U}_{\alpha_j}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{t,x}) \right| \right\} \leq L_V \mathbb{E} \left\{ \sup_{t \leq s \leq T} \|X_s^{(\alpha_j)t,x} - X_s^{t,x}\|_{\mathcal{H}} \right\}, \quad (5.70)$$

by Proposition 2.7. Hence it converges to zero as $j \rightarrow \infty$ by Proposition 3.1. For the second term in (5.69), we have

$$\begin{aligned} & \mathbb{E} \left\{ \left| \mathcal{U}_{\alpha_j}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{t,x}) - \mathcal{U}(\tau^* \wedge \tau_{\alpha_j}^*, X_{\tau^* \wedge \tau_{\alpha_j}^*}^{t,x}) \right| \right\} \leq \mathbb{E} \left\{ \sup_{t \leq s \leq T} |\mathcal{U}_{\alpha_j}(s, X_s^{t,x}) - \mathcal{U}(s, X_s^{t,x})| \right\} \\ & \leq \mathbb{E} \left\{ \sup \{ |\mathcal{U}_{\alpha_j}(s, y) - \mathcal{U}(s, y)|, (s, y) \in [t, T] \times \mathcal{K}_{t,T}^x(\omega) \} \right\} \end{aligned} \quad (5.71)$$

where $\mathcal{K}_{t,T}^x(\omega) := \{y : y = X_s^{t,x}(\omega), s \in [t, T]\}$ is a compact subset of \mathcal{H} in which the process $s \mapsto X_s^{t,x}(\omega)$ ranges. Take the limit as $j \rightarrow \infty$ and use dominated convergence to pass the limit inside the expectation. Hence (5.71) goes to zero as $j \rightarrow \infty$ since $\mathcal{U}_{\alpha_j} \rightarrow \mathcal{U}$ uniformly on compact subsets of $[0, T] \times \mathcal{H}$ (cf. Theorem 3.3). Finally the third term on the right hand side of (5.69) converges to zero by dominated convergence and continuity of the value function (cf. Theorem 3.3).

On the other hand, the left hand side of (5.68) converges pointwisely to \mathcal{U} by Theorem 3.3. Then taking the limit in (5.68) along the subsequence of Lemma 5.13 gives

$$\mathcal{U}(t, x) = \mathbb{E} \left\{ \mathcal{U}(\tau^*, X_{\tau^*}^{t,x}) \right\} = \mathbb{E} \left\{ \Theta(\tau^*, X_{\tau^*}^{t,x}) \right\}, \quad (5.72)$$

hence τ^* is optimal. \square

Appendices

A Proof of Lemma 3.7

By using the triangular inequality, Hölder inequality and the fact that τ is a stopping time, it is not hard to prove that

$$\begin{aligned} I_{\{s>\tau\}} \|X_s^{(\alpha)t,x;n} - X_\tau^{(\alpha)t,x;n}\|_{\mathcal{H}}^2 &\leq 3 I_{\{s>\tau\}} \left[(T-t) \int_t^s I_{\{\tau \leq u\}} \|A_{\alpha,n} X_u^{(\alpha)t,x;n}\|_{\mathcal{H}}^2 du \right. \\ &\quad \left. + \left\| \int_t^s I_{\{\tau \leq u\}} \sigma^{(n)}(X_u^{(\alpha)t,x;n}) dW_u^0 \right\|_{\mathcal{H}}^2 + \epsilon_n^2 \sum_{i=1}^n |W_s^i - W_\tau^i|^2 \right]. \end{aligned}$$

Take the supremum over all $s \in [t, T]$ and exploit the isometry of the stochastic integral; by the sublinear growth of σ and the bound on A_α , there exists a suitable constant $\gamma_{\alpha,n,T} > 0$ such that

$$\mathbb{E} \left\{ \sup_{t \leq s \leq T} \|X_s^{(\alpha)t,x;n} - X_\tau^{(\alpha)t,x;n}\|_{\mathcal{H}}^2 I_{\{s>\tau\}} \right\} \leq \gamma_{\alpha,n,T} \mathbb{E} \left\{ \int_t^T I_{\{\tau \leq u\}} (1 + \|X_u^{(\alpha)t,x;n}\|_{\mathcal{H}}^2) du \right\}. \quad (\text{A-1})$$

Using Hölder inequality twice in the right-hand side gives

$$\begin{aligned} \mathbb{E} \left\{ \int_t^T I_{\{\tau \leq u\}} (1 + \|X_u^{(\alpha)t,x;n}\|_{\mathcal{H}}^2) du \right\} &\leq \mathbb{E} \left\{ \left(\int_t^T I_{\{\tau \leq u\}} du \right)^{\frac{1}{2}} \left(\int_t^T (1 + \|X_u^{(\alpha)t,x;n}\|_{\mathcal{H}}^2)^2 du \right)^{\frac{1}{2}} \right\} \\ &\leq 2 \mathbb{E} \left\{ \left(\int_t^T I_{\{\tau \leq u\}} du \right)^{\frac{1}{2}} \right\} \mathbb{E} \left\{ \left(\int_t^T (1 + \|X_u^{(\alpha)t,x;n}\|_{\mathcal{H}}^4) du \right)^{\frac{1}{2}} \right\} \end{aligned}$$

and hence from (A-1) it follows

$$\mathbb{E} \left\{ \sup_{t \leq s \leq T} \|X_s^{(\alpha)t,x;n} - X_\tau^{(\alpha)t,x;n}\|_{\mathcal{H}}^2 I_{\{s>\tau\}} \right\} \leq \gamma_{\alpha,n,T} \sqrt{2T} \mathbb{E} \left\{ [T - \tau]^+ \right\}^{\frac{1}{2}} \mathbb{E} \left\{ 1 + \sup_{t \leq u \leq T} \|X_u^{(\alpha)t,x;n}\|_{\mathcal{H}}^4 \right\}^{\frac{1}{2}}. \quad (\text{A-2})$$

Now (3.9) is a consequence of (2.8) and (A-2).

B Finite dimensional variational inequality in bounded domains

Denote by $C_c^2(\mathbb{R}^n)$ the set of all C^2 -functions on \mathbb{R}^n with compact support. Recall the finite dimensional SDE (3.5). Let $n \in \mathbb{N}$ and $\alpha > 0$ be arbitrary but fixed. Let \mathcal{O}_R be the open ball in \mathbb{R}^n , centered in the origin and with radius R . Define $\tau_R(t, x)$ to be the first exit time from \mathcal{O}_R , i.e.

$$\tau_R(t, x) := \inf\{s \geq t : X_s^{(\alpha)t,x;n} \notin \mathcal{O}_R\} \wedge T. \quad (\text{B-1})$$

We are slightly abusing the notation by considering $\mathcal{H}^{(n)} \sim \mathbb{R}^n$ and $X^{(\alpha)t,x;n} \in \mathbb{R}^n$. For simplicity we set $\tau_R := \tau_R(t, x)$ and we introduce the optimal stopping problem arrested at τ_R ,

$$\mathcal{U}_{\alpha,R}^{(n)}(t, x^{(n)}) := \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ \Theta^{(n)}(\tau \wedge \tau_R, X_{\tau \wedge \tau_R}^{(\alpha)t,x;n}) \right\}. \quad (\text{B-2})$$

Proposition B.1. *For $(t, x^{(n)}) \in [0, T] \times \mathbb{R}^n$ fixed, $\mathcal{U}_{\alpha,R}^{(n)}(t, x^{(n)})$ converges to $\mathcal{U}_\alpha^{(n)}(t, x^{(n)})$ as $R \rightarrow \infty$. Moreover, the convergence is uniform on every compact subset $[0, T] \times \mathcal{K} \subset [0, T] \times \mathbb{R}^n$ and $\mathcal{U}_\alpha^{(n)} \in C([0, T] \times \mathbb{R}^n)$.*

Proof. Fix $(t, x^{(n)}) \in [0, T] \times \mathbb{R}^n$ and take $\bar{R} > 0$ such that $x^{(n)} \in \mathcal{O}_{\bar{R}}$; recall our notation $\tau_R := \tau_R(t, x)$. Now for all $R \geq \bar{R}$ we have

$$\begin{aligned} 0 \leq \mathcal{U}_{\alpha}^{(n)}(t, x^{(n)}) - \mathcal{U}_{\alpha, R}^{(n)}(t, x^{(n)}) &= \sup_{t \leq \sigma \leq T} \inf_{t \leq \tau \leq T} \mathbb{E} \left\{ \Theta^{(n)}(\sigma, X_{\sigma}^{(\alpha)t, x; n}) - \Theta^{(n)}(\tau \wedge \tau_R, X_{\tau \wedge \tau_R}^{(\alpha)t, x; n}) \right\} \\ &\leq \sup_{t \leq \sigma \leq T} \mathbb{E} \left\{ \left(\Theta^{(n)}(\sigma, X_{\sigma}^{(\alpha)t, x; n}) - \Theta^{(n)}(\sigma \wedge \tau_R, X_{\sigma \wedge \tau_R}^{(\alpha)t, x; n}) \right) I_{\{\sigma > \tau_R\}} \right\}, \end{aligned}$$

where $I_{\{\sigma > \tau_R\}}$ is the indicator function of the set $\{\omega \in \Omega : \sigma(\omega) > \tau_R(\omega)\}$. Hence by adding and subtracting $\Theta^{(n)}(\tau_R, X_{\tau_R}^{(\alpha)t, x; n})$ inside the expectation and by using the Lipschitz properties of $\Theta^{(n)}$ (cf. (2.5), (2.6)), we obtain

$$\begin{aligned} |\mathcal{U}_{\alpha, R}^{(n)}(t, x^{(n)}) - \mathcal{U}_{\alpha}^{(n)}(t, x^{(n)})| &\leq L'_{\Theta} C \mathbb{E} \left\{ \left(1 + \sup_{t \leq s \leq T} \|X_s^{(\alpha)t, x; n}\|_{\mathcal{H}}^p \right) [T - \tau_R]^+ \right\} \\ &\quad + L_{\Theta} \mathbb{E} \left\{ \sup_{t \leq s \leq T} \|X_s^{(\alpha)t, x; n} - X_{\tau_R}^{(\alpha)t, x; n}\|_{\mathcal{H}} I_{\{s > \tau_R\}} \right\}, \end{aligned}$$

with $p \geq 1$ as in Assumption 2.4. Therefore, for some positive constant $C_{\alpha, n, T} > 0$, we have

$$|\mathcal{U}_{\alpha, R}^{(n)}(t, x) - \mathcal{U}_{\alpha}^{(n)}(t, x)| \leq C_{\alpha, n, T} (1 + \|x\|_{\mathcal{H}} + \|x\|_{\mathcal{H}}^p) \left(\mathbb{E} \left\{ ([T - \tau_R]^+)^2 \right\}^{\frac{1}{2}} + \mathbb{E} \left\{ [T - \tau_R]^+ \right\}^{\frac{1}{4}} \right)$$

by Lemma 3.7 and the a priori estimates (2.8) and (2.9). Pointwise convergence now follows from dominated convergence and $[T - \tau_R]^+ \rightarrow 0$ as $R \rightarrow \infty$; in fact the right hand side decreases to zero.

Set

$$M_R(t, x) := (1 + \|x\|_{\mathcal{H}} + \|x\|_{\mathcal{H}}^p) \left(\mathbb{E} \left\{ ([T - \tau_R]^+)^2 \right\}^{\frac{1}{2}} + \mathbb{E} \left\{ [T - \tau_R]^+ \right\}^{\frac{1}{4}} \right)$$

and recall that, since the coefficients in (3.1) and (3.5) are time-homogeneous, the map $(t, x) \mapsto \tau_R(t, x)$ is \mathbb{P} -a.s. continuous (cf. [3], Lemma 3.2, p.332). Hence M_R is continuous in t, x for all $R > 0$. Now [13], Theorem 7.2.2 gives uniform convergence of $M_R(t, x)$ on compact subsets. Therefore

$$\lim_{R \rightarrow \infty} \sup_{(t, x^{(n)}) \in [0, T] \times \mathcal{K}} |\mathcal{U}_{\alpha, R}^{(n)}(t, x^{(n)}) - \mathcal{U}_{\alpha}^{(n)}(t, x^{(n)})| = 0$$

for every compact subset $\mathcal{K} \subset \mathbb{R}^n$. Since all $\mathcal{U}_{\alpha, R}^{(n)}$ are continuous, then $\mathcal{U}_{\alpha}^{(n)}$ is continuous on every compact subset $[0, T] \times \mathcal{K}$ and this is enough for global continuity in \mathbb{R}^n . \square

The infinitesimal generator of the diffusion $X^{(\alpha)x; n}$ is

$$\mathcal{L}_{\alpha, n} g := \frac{1}{2} \epsilon_n^2 \sum_{i=1}^n \frac{\partial^2 g}{\partial x_i^2} + \frac{1}{2} \sum_{i, j=1}^n [\sigma^{(n)} \sigma^{(n)*}]_{i, j} \frac{\partial^2 g}{\partial x_i \partial x_j} + \sum_{i=1}^n \left(\sum_{j=1}^n x_j \langle A_{\alpha} \varphi_j, \varphi_i \rangle \right) \frac{\partial g}{\partial x_i}, \quad (\text{B-3})$$

for $g \in C_c^2(\mathbb{R}^n)$. Notice that

$$[\sigma^{(n)} \sigma^{(n)*}]_{i, j}(x) = \langle \sigma^{(n)}(x), \varphi_i \rangle_{\mathcal{H}} \langle \sigma^{(n)}(x), \varphi_j \rangle_{\mathcal{H}}, \quad (\text{B-4})$$

since W^0 is a one dimensional Brownian motion. Moreover $\mathcal{L}_{\alpha, n}$ is a non-degenerate, uniformly elliptic operator. The bilinear form associated to the operator $\mathcal{L}_{\alpha, n}$ is

$$\begin{aligned} a_R^{(\alpha, n)}(u, w) &:= - \int_{\mathcal{O}_R} \mathcal{L}_{\alpha, n} u w \, dx = \frac{1}{2} \epsilon_n^2 \sum_{i=1}^n \int_{\mathcal{O}_R} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i} \, dx + \frac{1}{2} \sum_{i, j=1}^n \int_{\mathcal{O}_R} [\sigma^{(n)} \sigma^{(n)*}]_{i, j} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx \\ &\quad + \sum_{i, j=1}^n \int_{\mathcal{O}_R} \left(\frac{1}{2} \frac{\partial [\sigma^{(n)} \sigma^{(n)*}]_{i, j}}{\partial x_j} - x_j \langle A_{\alpha} \varphi_j, \varphi_i \rangle \right) \frac{\partial u}{\partial x_i} w \, dx, \end{aligned}$$

for $u, w \in H_0^1(\mathcal{O}_R)$. Denote by (\cdot, \cdot) the scalar product in $L^2(\mathcal{O}_R)$. From Assumption 2.2 and uniform ellipticity of $\mathcal{L}_{\alpha,n}$, it is not hard to see that there exist constants $\zeta_{\alpha,n,R}, C_{\alpha,n,R}, C'_{\alpha,n,R} > 0$ such that

$$|a_R^{(\alpha,n)}(u, w)| \leq C_{\alpha,n,R} \|u\|_{H_0^1(\mathcal{O}_R)} \|w\|_{H_0^1(\mathcal{O}_R)}, \quad (\text{B-5})$$

$$a_R^{(\alpha,n)}(u, u) + \zeta_{\alpha,n,R} (u, u) \geq C'_{\alpha,n,R} \|u\|_{H_0^1(\mathcal{O}_R)}^2. \quad (\text{B-6})$$

These properties guarantee well-posedness of the variational problem in the following proposition.

Define by $\mathcal{K}_{n,R}$ the n -dimensional closed convex set

$$\mathcal{K}_{n,R} := \left\{ w \in L^2(0, T; H_0^1(\mathcal{O}_R)) : \frac{\partial w}{\partial t} \in L^2(0, T; L^2(\mathcal{O}_R)), \text{ and } w \geq 0 \text{ a.e. in } (0, T) \times \mathcal{O}_R \right\}. \quad (\text{B-7})$$

Set

$$u_{\alpha,R}^{(n)} := \mathcal{U}_{\alpha,R}^{(n)} - \Theta^{(n)} \quad (\text{B-8})$$

and

$$\begin{aligned} f_{\alpha,n} &:= \frac{\partial \Theta^{(n)}}{\partial t} + \mathcal{L}_{\alpha,n} \Theta^{(n)} \\ &= \frac{\partial \Theta^{(n)}}{\partial t} + \frac{1}{2} \epsilon_n^2 \text{Tr} [D^2 \Theta^{(n)}] + \frac{1}{2} \text{Tr} [\sigma^{(n)} \sigma^{(n)*} D^2 \Theta^{(n)}] + \langle A_{\alpha,n} x, D \Theta^{(n)} \rangle_{\mathcal{H}}. \end{aligned} \quad (\text{B-9})$$

Usually the value function of an optimal stopping problem may be characterized as the solution of an obstacle problem in which the obstacle is the gain function. At this point we do not know the regularity of the function $\mathcal{U}_{\alpha,R}^{(n)}$ but, since $\Theta^{(n)}$ is smooth and $\mathcal{L}_{\alpha,n}$ is uniformly elliptic, we expect $\mathcal{U}_{\alpha,R}^{(n)}$ to solve the corresponding obstacle problem in the almost everywhere sense (cf. [17]). In other words the function $u_{\alpha,R}^{(n)} := \mathcal{U}_{\alpha,R}^{(n)} - \Theta^{(n)}$ should solve the obstacle problem with null obstacle. The regularity of the obstacle, (B-5) and (B-6) are sufficient to apply [3], Chapter 3, Theorems 2.2, 2.13, Corollaries 2.2, 2.3, 2.4 to obtain

Proposition B.2. *There exists a unique solution \bar{u} of the variational problem:*

$$\left\{ \begin{array}{l} u \in \mathcal{K}_{n,R}; \quad u(T, x^{(n)}) = 0, \quad x^{(n)} \in \overline{\mathcal{O}}_R; \\ -\left(\frac{\partial u}{\partial t}, w - u\right) + a_R^{(\alpha,n)}(u, w - u) - (f_{\alpha,n}, w - u) \geq 0 \quad \text{a.e. } t \in [0, T] \end{array} \right. \quad (\text{B-10})$$

for all $w \in \mathcal{K}_{n,R}$.

Moreover, $\bar{u} \in L^p(0, T; W_0^{1,p}(\mathcal{O}_R)) \cap L^p(0, T; W^{2,p}(\mathcal{O}_R))$, $\frac{\partial \bar{u}}{\partial t} \in L^p(0, T; L^p(\mathcal{O}_R))$ for all $1 \leq p < \infty$ and $\bar{u} \in C([0, T] \times \overline{\mathcal{O}}_R)$.

Corollary B.3. *The function \bar{u} coincides with the function $u_{\alpha,R}^{(n)}$ and uniquely solves the obstacle problem*

$$\left\{ \begin{array}{l} \max \left[\frac{\partial u}{\partial t} + \mathcal{L}_{\alpha,n} u + f_{\alpha,n}, -u \right] (t, x^{(n)}) = 0, \quad (t, x^{(n)}) \in (0, T) \times \mathcal{O}_R, \\ u(t, x^{(n)}) \geq 0 \quad \text{on } [0, T] \times \overline{\mathcal{O}}_R; \quad u(T, x^{(n)}) = 0, \quad x^{(n)} \in \overline{\mathcal{O}}_R; \\ u(t, x^{(n)}) = 0, \quad (t, x^{(n)}) \in [0, T] \times \partial \mathcal{O}_R; \end{array} \right. \quad (\text{B-11})$$

in the almost everywhere sense. Moreover the optimal stopping time for $\mathcal{U}_{\alpha,R}^{(n)}$ of (B-2) is

$$\tau_{\alpha,n,R}^* := \inf\{s \geq t : \mathcal{U}_{\alpha,R}^{(n)}(s, X_s^{(\alpha)t,x;n}) = \Theta^{(n)}(s, X_s^{(\alpha)t,x;n})\} \wedge \tau_R \wedge T \quad (\text{B-12})$$

and for any $\tau \leq \tau_{\alpha,n,R}^*$

$$\mathcal{U}_{\alpha,R}^{(n)}(t, x) = \mathbb{E} \left\{ \mathcal{U}_{\alpha,R}^{(n)}(\tau, X_\tau^{(\alpha)t,x;n}) \right\}. \quad (\text{B-13})$$

The proof follows from Proposition B.2 and is outlined in appendix C for completeness.

Now, in the spirit of [3],[27], [30], we obtain a variational problem for $u_{\alpha,R}^{(n)}$ weaker than (B-10) and (B-11).

Theorem B.4. *The function $u_{\alpha,R}^{(n)}$ is the unique solution of the weak variational problem:*

$$\left\{ \begin{array}{l} u \in L^2(0, T; H_0^1(\mathcal{O}_R)); \quad u(T, x^{(n)}) = 0, \quad x^{(n)} \in \overline{\mathcal{O}}_R; \\ u(t, x^{(n)}) \geq 0, \quad (t, x^{(n)}) \in [0, T] \times \overline{\mathcal{O}}_R; \\ \int_0^T \left[-\left(\frac{\partial w}{\partial t}, w - u\right) + a_R^{(\alpha,n)}(u, w - u) - (f_{\alpha,n}, w - u) \right] dt + \frac{1}{2} \|w(T)\|_{L^2(\mathcal{O}_R)}^2 \geq 0 \end{array} \right. \quad (\text{B-14})$$

for all $w \in \mathcal{K}_{n,R}$.

Proof. Since $u_{\alpha,R}^{(n)}$ is the solution of the stronger variational problems (B-10) and (B-11) it is not hard to show that it also solves (B-14) (cf. [3], p. 237). It remains to prove uniqueness. For that we need coerciveness condition (B-6). For simplicity set $\zeta := \zeta_{\alpha,n,R}$, $u_\zeta := e^{\zeta t} u_{\alpha,R}^{(n)}$, $f_\zeta := e^{\zeta t} f_{\alpha,n}$. Notice that $u_{\alpha,R}^{(n)}$ is the unique solution of (B-10) if and only if u_ζ is the unique solution of the same problem with $f_{\alpha,n}$ replaced by f_ζ and $a_R^{(\alpha,n)}(u, w)$ replaced by $a_\zeta(u, w) := a_R^{(\alpha,n)}(u, w) + \zeta(u, w)$. That also implies that u_ζ solves problem (B-14) when $a_R^{(\alpha,n)}(u, w)$ is replaced by $a_\zeta(u, w)$ and $f_{\alpha,n}$ is replaced by f_ζ . Then to prove uniqueness in problem (B-14) is equivalent to show that u_ζ is the unique solution of

$$\left\{ \begin{array}{l} u \in L^2(0, T; H_0^1(\mathcal{O}_R)); \quad u(T, x^{(n)}) = 0, \quad x^{(n)} \in \overline{\mathcal{O}}_R; \\ u(t, x^{(n)}) \geq 0, \quad (t, x^{(n)}) \in [0, T] \times \overline{\mathcal{O}}_R; \\ \int_0^T \left[-\left(\frac{\partial w}{\partial t}, w - u\right) + a_\zeta(u, w - u) - (f_\zeta, w - u) \right] dt + \frac{1}{2} \|w(T)\|_{L^2(\mathcal{O}_R)}^2 \geq 0 \end{array} \right. \quad (\text{B-15})$$

for all $w \in \mathcal{K}_{n,R}$.

Let \hat{v} be another solution of (B-15) and take $w = u_\zeta$ to obtain

$$\int_0^T \left[-\left(\frac{\partial u_\zeta}{\partial t}, u_\zeta - \hat{v}\right) + a_\zeta(t; \hat{v}, u_\zeta - \hat{v}) - (f_\zeta, u_\zeta - \hat{v}) \right] dt \geq 0, \quad (\text{B-16})$$

since $u_\zeta(T) = 0$ a.e. on \mathcal{O}_R . On the other hand, simple calculations allow us to write (B-10) as

$$\int_0^T \left[-\left(\frac{\partial u_\zeta}{\partial t}, w - u_\zeta\right) + a_\zeta(u_\zeta, w - u_\zeta) - (f_\zeta, w - u_\zeta) \right] dt \geq 0 \quad (\text{B-17})$$

for all $w \in L^2(0, T; H_0^1(\mathcal{O}_R))$ with $w \geq 0$. In (B-17) take $w = \hat{v}$ and add it to (B-16) to conclude

$$\int_0^T a_\zeta(\hat{v} - u_\zeta, \hat{v} - u_\zeta) dt \leq 0.$$

Now coerciveness (cf. (B-6)) and the definition of $a_\zeta(\cdot, \cdot)$ provide uniqueness. \square

Now for each $R > 0$, we consider the zero extension outside \mathcal{O}_R of $u_{\alpha, R}^{(n)}$ but we still denote it by $u_{\alpha, R}^{(n)}$ for simplicity.

Proposition B.5. *The family $(u_{\alpha, R}^{(n)})_{R>0}$ is uniformly bounded in $L^2(0, T; W^{1,2}(\mathbb{R}^n, \mu_n))$ and its bound is also independent of α and n .*

Proof. By arguments similar to those in the proof of Proposition 2.7 we have that $(\mathcal{U}_{\alpha, R}^{(n)})_{R>0}$ is a family uniformly bounded by $\bar{\Theta}$ and it is uniformly Lipschitz with respect to the space variable, with Lipschitz constant $L_{\mathcal{U}}$. It follows that the family $(u_{\alpha, R}^{(n)})_{R>0}$ is also uniformly bounded with bound $2\bar{\Theta}$ and equi-Lipschitz with constant $L_{\Theta} + L_{\mathcal{U}}$. From the latter we obtain $\|Du_{\alpha, R}^{(n)}(t)\|_{L^2(\mathbb{R}^n, \mu_n; \mathbb{R}^n)} \leq L_{\Theta} + L_{\mathcal{U}}$ and so

$$\int_0^T \|u_{\alpha, R}^{(n)}(t)\|_{W^{1,2}(\mathbb{R}^n, \mu_n)}^2 dt \leq C_{\mathcal{U}} T, \quad (\text{B-18})$$

with $C_{\mathcal{U}} = 4\bar{\Theta}^2 + (L_{\Theta} + L_{\mathcal{U}})^2$. \square

From Propositions B.1 and B.5 we simply obtain the next corollary.

Corollary B.6. *There exists a subsequence $(u_{\alpha, R_i}^{(n)})_{i \in \mathbb{N}}$ such that $R_i \rightarrow \infty$ as $i \rightarrow \infty$ and the function $u_{\alpha, R_i}^{(n)}$ converges to $u_{\alpha}^{(n)}$ as $i \rightarrow \infty$, weakly in $L^2(0, T; W^{1,2}(\mathbb{R}^n, \mu_n))$ and strongly in $L^2(0, T; L^p(\mathbb{R}^n, \mu_n))$, $1 \leq p < \infty$. Moreover $u_{\alpha}^{(n)}$ is identified as $u_{\alpha}^{(n)} = \mathcal{U}_{\alpha}^{(n)} - \Theta^{(n)}$.*

C Proof of Proposition B.3

By the regularity of \bar{u} in Proposition B.2, expression (B-10) is equivalent to

$$\left(\frac{\partial \bar{u}}{\partial t} + \mathcal{L}_{\alpha, n} \bar{u} + f, w - \bar{u} \right) \leq 0 \quad \text{a.e. } t \in [0, T]. \quad (\text{C-1})$$

Let $\zeta \in C_c^\infty(\mathcal{O}_R)$, $\zeta \geq 0$ and choose $w = \bar{u} + \zeta$ to obtain

$$\left(\frac{\partial \bar{u}}{\partial t} + \mathcal{L}_{\alpha, n} \bar{u} + f, \zeta \right) \leq 0, \quad \text{a.e. } t \in [0, T].$$

This clearly implies

$$\frac{\partial \bar{u}}{\partial t} + \mathcal{L} \bar{u} + f \leq 0, \quad \text{a.e. } \in [0, T] \times \bar{\mathcal{O}}_R. \quad (\text{C-2})$$

On the other hand, by choosing $w \equiv 0$, (C-1) reads

$$\left(\frac{\partial \bar{u}}{\partial t} + \mathcal{L}_{\alpha, n} \bar{u} + f, -\bar{u} \right) \leq 0, \quad \text{a.e. } t \in [0, T], \quad (\text{C-3})$$

and since $\bar{u} \geq 0$ it implies

$$\frac{\partial \bar{u}}{\partial t} + \mathcal{L}_{\alpha, n} \bar{u} + f \geq 0, \quad \text{a.e. on } [0, T] \times \bar{\mathcal{O}}_R. \quad (\text{C-4})$$

Now, (C-2) and (C-4) guarantee

$$\max \left[\frac{\partial \bar{u}}{\partial t} + \mathcal{L}_{\alpha,n} \bar{u} + f, -\bar{u} \right] = 0, \quad \text{a.e. } \in [0, T] \times \overline{\mathcal{O}}_R.$$

The regularity of $\partial \mathcal{O}_R$ and [1], Theorem 3.22 enable us to find a sequence $(u_j)_{j \in \mathbb{N}}$, such that $u_j \in C_c^\infty(\mathbb{R}^{n+1})$ and

$$\|u_j - \bar{u}\|_{W^{1,2,p}((0,T) \times \mathcal{O}_R)} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (\text{C-5})$$

In fact it suffices to take a partition of the domain and use the standard mollification on each element of the partition. Then (C-5) follows from the usual properties of the mollifiers and the fact that the operators ∂_t , D and D^2 are closed in L^p . Moreover, the continuity of \bar{u} and that of a suitable extension to \mathbb{R}^{n+1} imply that the convergence is also uniform on any compact set \mathcal{O}' such that $[0, T] \times \overline{\mathcal{O}}_R \subset \mathcal{O}'$; that is

$$\|u_j - \bar{u}\|_{L^\infty} \rightarrow 0, \quad \text{as } j \rightarrow \infty, \quad \text{on } \mathcal{O}'. \quad (\text{C-6})$$

Now we fix an arbitrary $t \in [0, T]$ and a stopping time $\tau \in [t, T]$. An application of Dynkin's formula from t to $\tau \wedge \tau_R$ gives

$$\mathbb{E} \left\{ u_j(\tau \wedge \tau_R, X_{\tau \wedge \tau_R}^{(\alpha)t,x;n}) \right\} = u_j(t, x^{(n)}) + \mathbb{E} \left\{ \int_t^{\tau \wedge \tau_R} \left(\frac{\partial u_j}{\partial s} + \mathcal{L}_{\alpha,n} u_j \right) (s, X_s^{(\alpha)t,x;n}) ds \right\}. \quad (\text{C-7})$$

On the other hand by [3], Chapter 2, Lemma 8.1 there exists a constant $C_{T,R} > 0$ such that

$$\left| \mathbb{E} \left\{ \int_t^{\tau \wedge \tau_R} \left(\frac{\partial}{\partial s} + \mathcal{L}_{\alpha,n} \right) (u_j - \bar{u}) (s, X_s^{(\alpha)t,x;n}) ds \right\} \right| \leq C_{T,R} \left\| \left(\frac{\partial}{\partial s} + \mathcal{L}_{\alpha,n} \right) (u_j - \bar{u}) \right\|_{L^2((0,T) \times \mathcal{O}_R)}, \quad (\text{C-8})$$

hence by taking the limit as $j \rightarrow \infty$ and by using (C-5) and (C-6) we obtain

$$\mathbb{E} \left\{ \bar{u}(\tau \wedge \tau_R, X_{\tau \wedge \tau_R}^{(\alpha)t,x;n}) \right\} = \bar{u}(t, x^{(n)}) + \mathbb{E} \left\{ \int_t^{\tau \wedge \tau_R} \left(\frac{\partial \bar{u}}{\partial s} + \mathcal{L}_{\alpha,n} \bar{u} \right) (s, X_s^{(\alpha)t,x;n}) ds \right\} \quad \text{for all } \tau \in [t, T]. \quad (\text{C-9})$$

Recall that (B-11) holds almost everywhere in $(0, T) \times \mathcal{O}_R$ and, being the diffusion uniformly non degenerate, the law of $X^{(\alpha)t,x;n}$ is absolutely continuous with respect to the Lebesgue measure on $(0, T) \times \mathcal{O}_R$. Then

$$\bar{u}(t, x^{(n)}) \geq \mathbb{E} \left\{ \int_t^{\tau \wedge \tau_R} f(s, X_s^{(\alpha)t,x;n}) ds \right\} \quad \text{for all } \tau \in [t, T]; \quad (\text{C-10})$$

in particular, with τ^* defined by

$$\tau^* := \inf \{ s \geq t : \bar{u}(s, X_s^{(\alpha)t,x;n}) = 0 \} \wedge \tau_R \wedge T, \quad (\text{C-11})$$

(B-11) implies

$$\bar{u}(t, x^{(n)}) = \mathbb{E} \left\{ \int_t^{\tau^*} f(s, X_s^{(\alpha)t,x;n}) ds \right\}. \quad (\text{C-12})$$

Therefore, by using (B-9) and by recalling (B-2) we have

$$\begin{aligned} \bar{u}(t, x^{(n)}) &= \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ \int_t^{\tau \wedge \tau_R} \left(\frac{\partial \Theta^{(n)}}{\partial s} + \mathcal{L}_{\alpha,n} \Theta^{(n)} \right) (s, X_s^{(\alpha)t,x;n}) ds \right\} \\ &= \sup_{t \leq \tau \leq T} \mathbb{E} \left\{ \Theta^{(n)}(\tau \wedge \tau_R, X_{\tau \wedge \tau_R}^{(\alpha)t,x;n}) \right\} - \Theta^{(n)}(t, x) = \mathcal{U}_{\alpha,R}^{(n)}(t, x) - \Theta^{(n)}(t, x). \end{aligned} \quad (\text{C-13})$$

It now follows that $\bar{u} = u_{\alpha,R}^{(n)}$ and $\tau^* = \tau_{\alpha,n,R}^*$.

Notice that for any stopping time $\tau \leq \tau_{\alpha,n,R}^*$, combining (C-13) and (C-9) gives

$$\mathcal{U}_{\alpha,R}^{(n)}(t, x^{(n)}) = \mathbb{E} \left\{ \mathcal{U}_{\alpha,R}^{(n)}(\tau, X_{\tau}^{(\alpha)t, x; n}) \right\}, \quad (\text{C-14})$$

i.e. the dynamic programming principle for $\mathcal{U}_{\alpha,R}^{(n)}$ holds.

D Proof of Lemma 5.3

Recall the definition

$$\phi^{(k)}(t, x) := \sum_{m=0}^{N_k} \left[a_m \cos(m t + \langle h_m, x \rangle_{\mathcal{H}}) + b_m \sin(m t + \langle h_m, x \rangle_{\mathcal{H}}) \right] \quad (\text{D-1})$$

and notice that

$$\frac{\partial \phi^{(k)}}{\partial t}(t, x) = \sum_{m=0}^{N_k} \left[-m a_m \sin(m t + \langle h_m, x \rangle_{\mathcal{H}}) + m b_m \cos(m t + \langle h_m, x \rangle_{\mathcal{H}}) \right]. \quad (\text{D-2})$$

The characteristic function of the centered Gaussian measure (see for example [9]) is

$$\int_{\mathcal{H}} e^{i \langle h, x \rangle_{\mathcal{H}}} \mu(dx) = e^{-\frac{1}{2} \langle Qh, h \rangle_{\mathcal{H}}}. \quad (\text{D-3})$$

Hence, direct computation gives

$$\|\phi^{(k)}\|_{W^{1,2}}^2 = \frac{2\pi}{T} \sum_{n=0}^{N_k} (1 + n^2)(a_n^2 + b_n^2), \quad (\text{D-4})$$

by using the exponential representation of trigonometric functions, simple trigonometric results and (D-3). Since $\phi^{(k)} \rightarrow w$ in $W^{1,2}([0, T] \times \mathcal{H}, \lambda \times \mu)$ as $k \rightarrow \infty$, then there exists a constant $C_w > 0$, only depending on w , such that $\|\phi^{(k)}\|_{W^{1,2}} < C_w$ for all k . This and (D-4) imply

$$\sum_{n=0}^{\infty} (1 + n^2)(a_n^2 + b_n^2) < \frac{T C_w}{2\pi}. \quad (\text{D-5})$$

Consider the continuous map $t \mapsto \|\phi^{(k)}(t)\|_{L^2(\mathcal{H}, \mu)}^2$, where

$$\begin{aligned} \|\phi^{(k)}(t)\|_{L^2(\mathcal{H}, \mu)}^2 &= \frac{1}{2} \sum_{n=0}^{N_k} \left[a_n^2 + b_n^2 + (a_n^2 - b_n^2) e^{-2 \langle Qh_n, h_n \rangle_{\mathcal{H}}} \cos(2nt) \right] \\ &\quad + \sum_{n \neq m}^{N_k} a_m b_n \frac{1}{2} \left[e^{-\frac{1}{2} \langle Q(h_n + h_m), (h_n + h_m) \rangle_{\mathcal{H}}} \sin((n+m)t) \right. \\ &\quad \left. + e^{-\frac{1}{2} \langle Q(h_n - h_m), (h_n - h_m) \rangle_{\mathcal{H}}} \sin((n-m)t) \right] \end{aligned} \quad (\text{D-6})$$

by (D-1) and (D-3). Now, from (D-5) and (D-6) we obtain

$$\sup_{0 \leq t \leq T} \|\phi^{(k)}(t)\|_{L^2(\mathcal{H}, \mu)}^2 \leq \sum_{n=0}^{\infty} (a_n^2 + b_n^2) + 2 \left(\sum_{n=0}^{\infty} a_n^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} b_n^2 \right)^{\frac{1}{2}} < C_{2,w} \quad (\text{D-7})$$

for all $k \in \mathbb{N}$ and a suitable constant $C_{2,w} > 0$ only depending on w and T .

From (D-2) and (D-3), and by arguments similar to those used to obtain (D-6), we derive

$$\begin{aligned} \sup_{0 \leq t \leq T} \left\| \frac{\partial \phi^{(k)}}{\partial t}(t) \right\|_{L^2(\mathcal{H}, \mu)}^2 &\leq 2 \sum_{n=0}^{\infty} n^2 (a_n^2 + b_n^2) \\ &+ \left(\sum_{n=0}^{\infty} n^2 a_n^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} n^2 b_n^2 \right)^{\frac{1}{2}} < C_{3,w} \end{aligned} \quad (\text{D-8})$$

for all $k \in \mathbb{N}$ and a suitable $C_{3,w} > 0$ depending only on w and T .

Therefore, it is not hard to see that

$$\|[\phi^{(k)}(t)]^+\|_{L^2(\mathcal{H}, \mu)}^2 \leq \|\phi^{(k)}(t)\|_{L^2(\mathcal{H}, \mu)}^2 < C_{2,w} \quad \text{for all } t \in [0, T], \quad (\text{D-9})$$

by (D-7). Also, since

$$\begin{aligned} &\left| \|[\phi^{(k)}(t, x)]^+\|^2 - \|[\phi^{(k)}(s, x)]^+\|^2 \right| \\ &= \left| [\phi^{(k)}(t, x)]^+ + [\phi^{(k)}(s, x)]^+ \right| \left| [\phi^{(k)}(t, x)]^+ - [\phi^{(k)}(s, x)]^+ \right| \\ &\leq \left| [\phi^{(k)}(t, x)]^+ + [\phi^{(k)}(s, x)]^+ \right| \left| \phi^{(k)}(t, x) - \phi^{(k)}(s, x) \right|, \end{aligned}$$

we have that

$$\begin{aligned} &\left| \|[\phi^{(k)}(t)]^+\|_{L^2(\mathcal{H}, \mu)}^2 - \|[\phi^{(k)}(s)]^+\|_{L^2(\mathcal{H}, \mu)}^2 \right| \\ &\leq \|[\phi^{(k)}(t)]^+ + [\phi^{(k)}(s)]^+\|_{L^2(\mathcal{H}, \mu)} \|\phi^{(k)}(t) - \phi^{(k)}(s)\|_{L^2(\mathcal{H}, \mu)} \\ &\leq 2\sqrt{C_{2,w}} \|\phi^{(k)}(t) - \phi^{(k)}(s)\|_{L^2(\mathcal{H}, \mu)} \\ &\leq 2\sqrt{C_{2,w}C_{4,w}} |t - s| \end{aligned}$$

for all $t, s \in [0, T]$, by (D-8).

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